Proof of Euler's Identity

Suppose $\underline{x} = (x_1, x_2, x_3)$ denotes the Eulerian coordinates of a fluid particle with Lagrangian coordinates $\underline{X} = (X_1, X_2, X_3)$ and moving with velocity $\underline{u} = (u_1, u_2, u_3)$.

Recall: $\underline{u} = \frac{D\underline{x}}{Dt}$

where $\frac{D}{Dt} = \frac{\partial}{\partial t}\Big|_X$

We have that

$$\frac{D}{Dt}\left(\frac{\partial x_n}{\partial X_i}\right) = \frac{\partial}{\partial X_i}\left(\frac{Dx_n}{Dt}\right) = \frac{\partial u_n}{\partial X_i} = \frac{\partial u_n}{\partial x_m}\frac{\partial x_m}{\partial X_i} \qquad (*)$$

Let J be the Jacobian of the transformation between Eulerian and Lagrangian coordinates,

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \varepsilon_{ijk}\frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k}$$

where $\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k \text{ are in cyclic order} \\ -1 & \text{if } i,j,k \text{ in a cyclic order} \\ 0 & \text{otherwise} \end{cases}$

So $$\frac{DJ}{Dt} = \frac{D}{Dt}\left(\varepsilon_{ijk}\frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k}\right)$$

Product rule

$$= \varepsilon_{ijk}\left\{\frac{D}{Dt}\left(\frac{\partial x_1}{\partial X_i}\right)\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{D}{Dt}\left(\frac{\partial x_2}{\partial X_j}\right)\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{D}{Dt}\left(\frac{\partial x_3}{\partial X_k}\right)\right\}$$

using $(*)$

$$= \varepsilon_{ijk}\left\{\frac{\partial u_1}{\partial x_m}\frac{\partial x_m}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial u_2}{\partial x_m}\frac{\partial x_m}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial u_3}{\partial x_m}\frac{\partial x_m}{\partial X_k}\right\}$$

$$\partial X_i \, \partial X_j \, \partial x_m \, \partial X_k)$$

distribute ε_{ijk} & use Jacobian definition

$$\Rightarrow \boxed{\frac{DJ}{Dt} = \frac{\partial u_1}{\partial x_m}\frac{\partial(x_m, x_2, x_3)}{\partial(X_1, X_2, X_3)} + \frac{\partial u_2}{\partial x_m}\frac{\partial(x_1, x_m, x_3)}{\partial(X_1, X_2, X_3)} + \frac{\partial u_3}{\partial x_m}\frac{\partial(x_1, x_2, x_m)}{\partial(X_1, X_2, X_3)}}$$

Now

$$\frac{\partial u_1}{\partial x_m}\frac{\partial(x_m, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \frac{\partial u_1}{\partial x_1}\frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} + \cancel{\frac{\partial u_1}{\partial x_2}\frac{\partial(x_2, x_2, x_3)}{\partial(X_1, X_2, X_3)}}^{=0} + \cancel{\frac{\partial u_1}{\partial x_3}\frac{\partial(x_3, x_2, x_3)}{\partial(X_1, X_2, X_3)}}^{=0}$$

Since determinants are zero if they have repeated rows.

$$\therefore \frac{DJ}{Dt} = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}$$

$$\boxed{\frac{DJ}{Dt} = (\nabla \cdot \underline{u})\, J}$$