Homological Algebra

30 July 2024

Notes by Yuhang Wei Lectures by Kobi Kremnitzer Partially Based on Previous Notes by Sebastian Monnet

Contents

Preface

These notes are mostly based on the University of Oxford course 'C2.2 Homological Algebra' lectured by Prof Kobi Kremnitzer during the 2023-24 Michaelmas term. Portions of the previous notes (Monnet and Kremnitzer 2021) are reused, along with relevant parts of (Weibel 1994), upon which the course is largely based. Another important reference is (Rotman 2009), which provides detailed and, at times, meticulous proofs. I have labelled the source of many proofs (whether they are similar or different to the ones presented) for the reader's reference.

An overview of these notes follows. Section 1 states without proof some results from **module theory** which we will use later. Section 2 is a crash course on **category theory**, based on which Section 3 constructs **abelian categories** in a step-by-step manner. In general, abelian categories serve as the 'stage' for homological algebra, but as we will see, R-**Mod**, the category of R-modules, which is 'concrete' and thus easier to work with, is in fact a sufficient representative of abelian categories. Focusing on R-**Mod**, we then move on to establish the **module tensor product** (Section 4), where the **tensor-hom adjunction** emerges as a significant result. We then prove that R-**Mod** has **enough projectives and injectives** in Section 5, a property crucial to constructing resolutions in R-**Mod**. Section 6 then discusses **(co)chain complexes** and their **(co)homology**, which originally arise in algebraic topology but are viewed solely as algebraic entities here; they lead to the definition and several important properties of **resolutions**. With all the tools in hand, in Section 7 we are able to define our main protagonist, **derived functors**, which are proven to be **homological** δ-**functors**, in some sense a generalisation of (co)homology functors. The two main **derived functors** we study are Ext and Tor, induced by Hom and tensor products respectively, as defined in Section 8. Ext and Tor possess a crucial property of being **balanced**, which requires the introduction of **mapping cones** and **double and total complexes** for proof. Further properties of Ext, including its **ring structure** and its connection with **module extensions**, are discussed in Section 9 and Section 12, while further properties of Tor, demonstrated by **flat modules** and the **Universal Coefficient Theorem**, are the topic of Section 10. The machinery we build is also applied to construct **Koszul (co)homology** and **group (co)homology** in Section 11 and Section 13, respectively.

For most of the proofs, I have tried to improve them by filling in more detailed steps by using available references and adding cross-references to previous results in the notes. Two large deviations from the lectures are Section 3 and Section 4. A lot more details are supplemented in both sections so as to make them as self-contained as possible. I have also chosen to introduce module tensor products based on (Rotman 2009), starting from the universal mapping problem of R-biadditive maps, whereas the lectures used the tensor product of vector spaces as an initial motivation.

Homological Algebra is admittedly a challenging yet rewarding course. On a personal note, I chose to work on these notes to enhance my own learning. As a learner, I acknowledge that these notes must contain mistakes and improvable parts. Therefore, the reader is welcome to submit issues for any advice on GitHub (https://github.com/EricWay1024/Homological-Algebra-Notes), where these notes are open-sourced and updated[1]. One can also find a not-so-colourful version of these notes fit for printing by following that link.

Finally, I would like to thank Prof Kobi Kremnitzer for delivering the lectures and Sebastian Monnet for creating the previous version of these notes. I would also like to thank my friends Qixuan Fang for reading these notes and offering feedback and Quanwen Chen for contributing to the GitHub workflows.

Yuhang Wei

[1]For anyone interested, I write these notes with Typst, a fairly new but much simpler alternative of LaTeX.

1. Module Theory Recap

Definition 1.1. Let R be a ring. A **left R-module** M is an abelian group with maps $R \times M \to M$ (called multiplication), denoted as $(r, m) \mapsto r \cdot m = rm$, which satisfies:

$$\begin{aligned} r(m_1 + m_2) &= rm_1 + rm_2, \\ (r_1 + r_2)m &= r_1 m + r_2 m, \\ (r_1 r_2)m &= r_1(r_2 m), \\ 1_R \cdot m &= m. \end{aligned}$$

A **right R-module** is defined similarly, but with multiplication on the right, namely mr.

If R is a commutative ring, then left and right R-modules are the same, and we call them **R-modules**.

Definition 1.2. Let M be a left R-module. A **submodule** N of M satisfies:
- N is a subgroup of $(M, +)$;
- $rn \in N$ for all $r \in R$ and $n \in N$.

In this case we denote $N \subset M$.

Definition 1.3. Let R be a ring. Let M_1, M_2 be left R-modules. A map $\varphi : M_1 \to M_2$ is a **module homomorphism** if it satisfies:

$$\begin{aligned} \varphi(x + y) &= \varphi(x) + \varphi(y), \\ \varphi(rx) &= r\varphi(x). \end{aligned}$$

for all $x, y \in M_1$ and $r \in R$.

Definition 1.4. The **kernel** of a module homomorphism $\varphi : M_1 \to M_2$ is defined as

$$\mathrm{Ker}(\varphi) \coloneqq \{x \in M_1 : \varphi(x) = 0\}.$$

The **image** of φ is defined as

$$\mathrm{Im}(\varphi) \coloneqq \{\varphi(x) : x \in M_1\}.$$

It can be shown that $\mathrm{Ker}(\varphi) \subset M_1$ and $\mathrm{Im}(\varphi) \subset M_2$.

Definition 1.5. Let $N \subset M$ be left R-modules. Define a left R-module on the quotient group M/N with

$$r(x + N) = rx + N$$

for all $r \in R$ and $x \in M$. Then the **quotient map** $M \to M/N$ is a module homomorphism and M/N is a **quotient module**.

Definition 1.6. Let X be a set. The **free module** with basis X is defined as

$$R^{\oplus X} = \bigoplus_{x \in X} Rx.$$

We have the inclusion map $i : X \to R^{\oplus X}$ between sets:

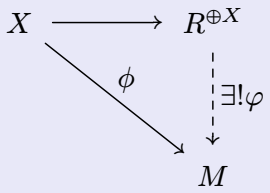
$$i(x) = 1_R \cdot x.$$

An element $m \in R^{\oplus X}$ can be written as

$$m = \sum_{x \in X} a_x x,$$

where only finitely many $a_x \in R$ is non-zero.

Proposition 1.7. For any R-module M and map between sets $\phi : X \to M$, there exists a unique module homomorphism $\varphi : R^{\oplus X} \to M$ that make the following commute:



Definition 1.8. Let X be a subset of R-module M and let $i : X \to M$ be the inclusion map. We have the corresponding map $\sigma : R^{\oplus X} \to M$. We say

- X is **linear independent** or **free** if σ is injective and X is **linear dependent** otherwise;
- X spans or generates M if σ is surjective, in which case X is a **generating set** of M. A module with a finite generating subset is called a **finitely generated module**.

A linear independent generating subset of M is called a **basis** of M, and a module with a basis is called a **free module**.

Corollary 1.9. Any R-module M is isomorphic to a quotient of a free module.

Proposition 1.10. Any submodule of a free module over a PID is free.

2. Basic Category Theory

This section is a crash course in category theory. The reader is advised to take the Category Theory course concurrently and/or refer to other materials, e.g. (Awodey 2010).

2.1. Basic Definitions

Definition 2.1.1. A **category** $\mathcal{C}$ consists of

- A collection of **objects** $\operatorname{ob} \mathcal{C}$ and
- For every pair of objects $X, Y \in \operatorname{ob} \mathcal{C}$, a collection of **morphisms** $\operatorname{Hom}_{\mathcal{C}}(X, Y)$, where for $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$ we denote $f : X \to Y$ or $X \xrightarrow{f} Y$ and say X is the **domain** of f and Y is the **codomain** of f;

such that

- For every object X, there exists an **identity morphism** $\operatorname{id}_X \in \operatorname{Hom}_{\mathcal{C}}(X, X)$;
- For every pair of morphisms $f : X \to Y$ and $g : Y \to Z$, there exists a **composite morphism** $g \circ f : X \to Z$,

subject to the axioms:

- For every morphism $f : X \to Y$, we have $\operatorname{id}_y \circ f = f \circ \operatorname{id}_X = f$;
- For every triple of morphisms $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, we have $(h \circ g) \circ f = h \circ (g \circ f)$, which we simply denote as $h \circ g \circ f$.

Notation 2.1.2. We usually write $X \in \mathcal{C}$ when we mean $X \in \operatorname{ob} \mathcal{C}$. We sometimes denote $\operatorname{Hom}_{\mathcal{C}}(X, X)$ as $\operatorname{End}_{\mathcal{C}}(X)$ (the **endomorphisms** of X). We might simply write Hom instead of $\operatorname{Hom}_{\mathcal{C}}$ if the underlying category is clear from the context.

Definition 2.1.3. A category $\mathcal{C}$ **locally small** if for every $X, Y \in \mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is a set. A category $\mathcal{C}$ is **small** if it is locally small and further $\operatorname{ob} \mathcal{C}$ is a set.

These definitions above are to avoid set-theoretic size issues, which we shall not delve into. They are employed when necessary to ensure that we do not run into paradoxes.

Example 2.1.4. A **discrete category** $\mathcal{C}$ is one where

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) = \begin{cases} \{\operatorname{id}_X\} & X = Y \\ \emptyset & X \neq Y \end{cases}$$

It does not contain more information than $\operatorname{ob} \mathcal{C}$, so it can be simply regarded as a collection of objects, or a set when $\mathcal{C}$ is small.

Example 2.1.5. If $\operatorname{ob} \mathcal{C} = \{x\}$, then $\operatorname{Hom}_{\mathcal{C}}(x, x)$ is a **monoid**.

If you have never heard of monoids before, the above can be seen as the definition of a monoid. In general, a category is a "generalised" monoid because in a category you can only compose two mor-

phisms f, g in certain situations (namely, when the codomain of f and the domain of g match), whereas composition is allowed for any two elements of a monoid.

Example 2.1.6. The following are the main categories we will be working with.

- The category $\mathbf{Set}$ has objects which are sets and morphisms which are functions between sets. Notice in category theory we avoid talking directly about elements of a set, because a set, which is an object of the category $\mathbf{Set}$, is "atomic" or inseparable.
- Let k be a field. The category $\mathbf{Vect}_k$ has objects which are vector spaces over k and morphisms which are linear transformations between vector spaces. We often denote $\mathrm{Hom}_{\mathbf{Vect}_k}$ as Hom_k. In particular, for any $V, W \in \mathbf{Vect}_k$, $\mathrm{Hom}_k(V, W)$ is also a vector space.
- Let R be a ring. The category R-$\mathbf{Mod}$ has objects which are left R-modules and morphisms which are module homomorphisms. Similarly, we have the category $\mathbf{Mod}$-R of right R-modules. We often denote $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}$ or $\mathrm{Hom}_{\mathbf{Mod}\text{-}R}$ as Hom_R; it should be clear from the context which one we are referring to.
- The category $\mathbf{Grp}$ has objects which are groups and morphisms which are group homomorphisms. Similarly, we have the category $\mathbf{Ab}$ of abelian groups.

Definition 2.1.7. Let $\mathcal{C}, \mathcal{D}$ be categories. The **product category** $\mathcal{C} \times \mathcal{D}$ consists of objects (C, D) for $C \in \mathcal{C}$ and $D \in \mathcal{D}$, and morphisms $(f, g) : (C_1, D_1) \to (C_2, D_2)$ for $f : C_1 \to C_1$ and $g : D_1 \to D_2$.

Definition 2.1.8. A morphism $f : B \to C$ is **monic** (or a **monomorphism**) if for any $e_1, e_2 : A \to B$ such that $f \circ e_1 = f \circ e_2$ we have $e_1 = e_2$. A morphism $f : B \to C$ is **epic** (or an **epimorphism**) if for any $g_1, g_2 : C \to D$ such that $g_1 \circ f = g_2 \circ f$ we have $g_1 = g_2$.

Note 2.1.9. $f : B \to C$ is monic if and only if the induced map $(f \circ -) : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ is injective for any A, and $f : B \to C$ is epic if and only if the induced map $(- \circ f) : \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Hom}_{\mathcal{C}}(B, D)$ is injective for any D.

Example 2.1.10. In $\mathbf{Set}$, a monomorphism is equivalent to a one-to-one map and an epimorphism is equivalent to an onto map.

Example 2.1.11. In the category of commutative rings, $\mathbb{Z} \to \mathbb{Q}$ is both monic and epic. Note that if two maps agree on $\mathbb{Z} \to R$, they must also agree on $\mathbb{Q} \to R$, since a ring homomorphism $f : \mathbb{Q} \to R$ is uniquely determined by $f(1)$.

Example 2.1.12. In the category of commutative rings, for any ring R and its ideal I, $R \to R/I$ is epic.

2.2. Categories with a Zero Object

Definition 2.2.1. An **initial object** I of category $\mathcal{C}$ is an object such that for any $A \in \text{ob } \mathcal{C}$, there exists a unique morphism $I \to A$.

A **final object** T is an object such that for any $A \in \text{ob } \mathcal{C}$ there exists a unique morphism $A \to T$.

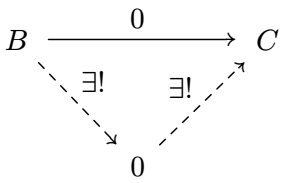
Example 2.2.2. In $\mathbf{Set}$, an initial object is equivalent to an empty set, while a final object is equivalent to a one-element (or singleton) set.

Definition 2.2.3. A **zero object** 0 is both initial and final.

Example 2.2.4. In R-$\mathbf{Mod}$, a zero object is equivalent to the zero module.

Proposition 2.2.5. If there is a zero object in the category, then for any $B, C \in \mathcal{C}$ we have a **zero morphism** $0 \in \text{Hom}_{\mathcal{C}}(B, C)$ which factors through the zero object.

Proof.

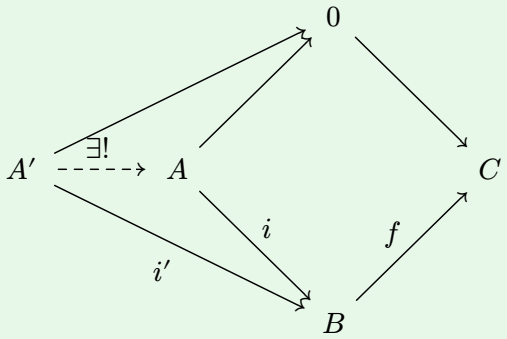


It is clear from the commutative diagram. ■

Notation 2.2.6. In a commutative diagram, two paths with the same starting and ending points correspond to two equal morphisms.

Notation 2.2.7. We (ab)use the notation 0 to denote both a zero object and a zero morphism.

Definition 2.2.8. In a category with a zero object, a **kernel** of $f : B \to C$ is a morphism $i : A \to B$ such that $f \circ i = 0$ in a universal way. That is, for any $i' : A' \to B$ such that $f \circ i' = 0$, there exists a unique morphism $h : A' \to A$ such that $i' = i \circ h$. We denote $i = \ker(f)$. Diagrammatically,



Notation 2.2.9. Sometimes, people might also say the object A in the above definition is the kernel of f when the morphism i is clear, and write $A = \ker(f)$. However, this easily leads to confusion later on, so this note adapts the following non-standard notation: we write $A = \mathrm{Ker}(f)$ (with a capital K) when we mean the object and $i = \ker(f)$ when we mean the morphism. Hence, we would have

$$\mathrm{Ker}(f) \overset{\ker(f)}{\longrightarrow} B \overset{f}{\to} C$$

such that $f \circ \ker(f) = 0$ in a universal way. Similar notations will be used for concepts we define later.

Example 2.2.10. In $\mathbf{Vect}_k$, kernels are kernels.

Theorem 2.2.11. A kernel is a monomorphism.

Definition 2.2.12. A **cokernel** of $f : B \to C$ is a morphism $j : C \to D$ such that $j \circ f = 0$ in a universal way. We denote $j = \mathrm{coker}(f)$ and $D = \mathrm{Coker}(f)$.

Theorem 2.2.13. A cokernel is an epimorphism.

Example 2.2.14. In $\mathbf{Vect}_k$, the cokernel of $T : V \to W$ is the quotient map $W \to W/\operatorname{im} T$.

Lemma 2.2.15. Let A be any object. Then for the unique morphism $f : A \to 0$, we have $\ker(f) = \mathrm{id}_A$ and $\operatorname{coker}(f) = \mathrm{id}_0 = 0$. Dually, for $g : 0 \to A$, we have $\ker(g) = 0$ and $\operatorname{coker}(g) = \mathrm{id}_A$.

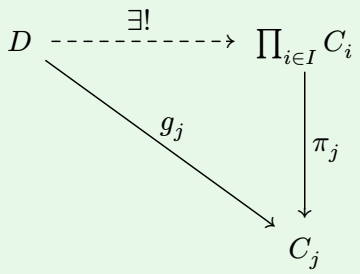
Definition 2.2.16. The **opposite category** of $\mathcal{C}$ is a category $\mathcal{C}^{\mathrm{op}}$ where $\operatorname{ob} \mathcal{C}^{\mathrm{op}} = \operatorname{ob} \mathcal{C}$ and $\operatorname{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) = \operatorname{Hom}_{\mathcal{C}}(y, x)$.

Proposition 2.2.17. A morphism $f : B \to C$ is monic in $\mathcal{C}$ if and only if $f^{\mathrm{op}} : C \to B$ is epic in $\mathcal{C}^{\mathrm{op}}$.

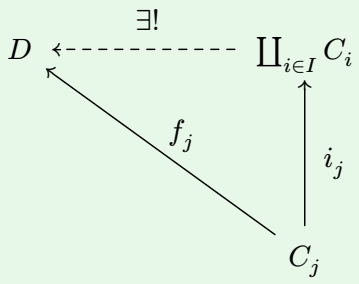
We say that "monic" and "epic" are **dual** concepts. Similarly, "initial objects" and "final objects" are dual; "kernels" and "cokernels" are dual.

2.3. Products and Coproducts

Definition 2.3.1. Let $\{C_i \mid i \in I\}$ be a family of objects, then their **product** $\prod_{i \in I} C_i$ is an object such that there exist $\pi_j : \prod_{i \in I} \to C_j$ for all $j \in I$ in a universal way. That is, for any object D with morphisms $g_j : D \to C_j$ for all $j \in I$, there exists a unique morphism $D \to \prod_{i \in I}$.



The **coproduct** of $\{C_i \mid i \in I\}$ is defined as their product in the opposite category C^{op}.



Example 2.3.2. In **Set**, let $\{X_i \mid i \in I\}$ be a family of sets.

$$\prod_{i\in I} X_i = \left\{(x_i)_{i\in I} \mid x_i \in X_i\right\}$$

and $\coprod_{i\in I} X_i$ is the disjoint union.

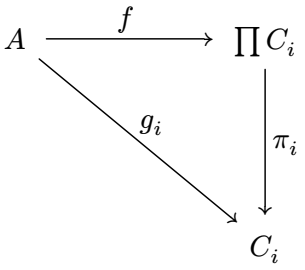
Remark 2.3.3. We need to use tuples here for the ordering of elements; suppose we want to use sets only, then it can be messy and arbitrary! This is an advantage of the language of category theory over that of set theory.

Proposition 2.3.4.

$$\mathrm{Hom}_{\mathcal{C}}\left(A, \prod C_i\right) \xrightarrow{\sim} \prod \mathrm{Hom}_{\mathcal{C}}(A, C_i)$$

Proof. For any C_i, there exists $\pi_i : \prod C_i \to C_i$ satisfying the universal property. Define $\varphi : \mathrm{Hom}_{\mathcal{C}}(A, \prod C_i) \to \prod \mathrm{Hom}_{\mathcal{C}}(A, C_i)$ as

$$f \mapsto (\pi_i \circ f)_i = (\pi_1 \circ f, ..., \pi_n \circ f)$$

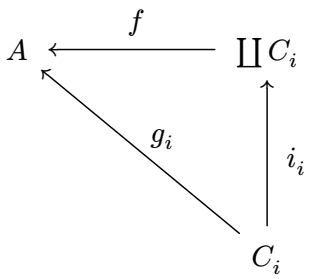


Any $(g_i)_i \in \prod \mathrm{Hom}_{\mathcal{C}}(A, C_i)$ can be factorised as $(\pi_i \circ f')_i$ for some unique $f' : A \to \prod C_i$ due to the universal property of the product. The existence of f' ensures that φ is surjective and the uniqueness of f ensures injectivity. Thus φ is a bijection. ■

Proposition 2.3.5. We have

$$\mathrm{Hom}_{\mathcal{C}}\left(\coprod C_i, A\right) \xrightarrow{\sim} \prod \mathrm{Hom}_{\mathcal{C}}(C_i, A).$$

Proof. This is similar to the above case: we just reverse all the arrows.



Notice the asymmetry here. It is not coproduct on the right hand side because it is still a tuple of arrows. ■

2.4. Functors and Natural Transformations

Definition 2.4.1. Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of

- A map of objects $\mathrm{ob}\ \mathcal{C} \to \mathrm{ob}\ \mathcal{D}$;
- For every pair objects $C_1, C_2 \in \mathcal{C}$, a map of morphisms

$$\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{D}}(F(C_1), F(C_2))$$

subject to preserving morphism composition and identity morphisms.

Definition 2.4.2. Now we can define $\mathbf{Cat}$, the category of all (small) categories, where $\mathrm{ob}\ \mathbf{Cat}$ are small categories and $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ are functors between $\mathcal{C}$ and $\mathcal{D}$.

Definition 2.4.3. Suppose $F, G : \mathcal{C} \to \mathcal{D}$, then a **natural transformation** $\alpha : F \Rightarrow G$ is defined by a collection of morphisms in $\mathcal{D}$ indexed by $x \in \mathrm{ob}\ \mathcal{C}$:

$$\{\alpha_x : F(x) \to G(x)\}_{x \in \mathrm{ob}\ \mathcal{C}}$$

where the diagram commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{\alpha_x} & G(x) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(x') & \xrightarrow{\alpha_{x'}} & G(x') \end{array}$$

Definition 2.4.4. The **functor category** $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is a category where the objects are functors $\mathcal{C} \to \mathcal{D}$ and the morphisms are natural transformations.

Remark 2.4.5. In $\mathbf{Cat}$, the hom-sets are not only sets but also categories, which means that $\mathbf{Cat}$ is a **2-category**.

2.5. Adjoint Functors

Definition 2.5.1. Functors $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$ are **adjoint** if for all $A \in \mathcal{A}, B \in \mathcal{B}$ there exists a bijection

$$\tau_{AB} : \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(A, R(B))$$

such that for any $f : A \to A'$ and $g : B \to B'$, the diagram commutes:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{B}}(L(A'), B)) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(L(A), B) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(L(A), B') \\ \downarrow{\scriptstyle \tau_{A'B}} & & \downarrow{\scriptstyle \tau_{AB}} & & \downarrow{\scriptstyle \tau_{AB'}} \\ \mathrm{Hom}_{\mathcal{A}}(A', R(B)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, R(B')) \end{array}$$

Remark 2.5.2. Recall in linear algebra we have $\langle Tv, w\rangle = \langle v, T^* w\rangle$, where the name "adjoint" comes from.

Remark 2.5.3. Equivalently, τ is a natural isomorphism between $\mathrm{Hom}_{\mathcal{B}}(L(-), -)$ and $\mathrm{Hom}_{\mathcal{A}}(-, R(-))$, both of which are functors $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathbf{Set}$. Note that A^{op} is used here because $\mathrm{Hom}_{\mathcal{A}}(-, B)$ is a contravariant functor.

Example 2.5.4. Free is the left adjoint of Forget. For example, we define the functors between $\mathbf{Vect}_k$ and $\mathbf{Set}$:

$$\text{Forget} : \mathbf{Vect}_k \to \mathbf{Set}$$
$$(V, +, \cdot) \mapsto V$$

$$\text{Free} : \mathbf{Set} \to \mathbf{Vect}_k$$
$$X \mapsto k[X]$$

Then we have:

$$\text{Hom}_{\mathbf{Vect}_k}(k[X], W) \cong \text{Hom}_{\mathbf{Set}}(X, \text{Forget}(W))$$
$$T \mapsto T|_X$$
$$\text{linearly extended } f \hookleftarrow f$$

2.6. Equivalence of Categories

Definition 2.6.1. In a category $\mathcal{C}$, objects X, Y are **isomorphic** if there exists $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$. We say that f and g are **isomorphisms**.

In the functor category, an isomorphism (which is a natural transformation between functors) is often called a **natural isomorphism**.

Consider $\mathbf{Cat}$, then two small categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G = \text{Id}$ and $G \circ F = \text{Id}$. However, this rarely happens. We hence introduce the following weaker condition.

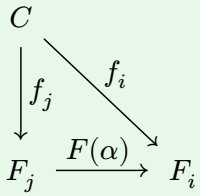
Definition 2.6.2. Two categories $\mathcal{C}$ and $\mathcal{D}$ are **equivalent** if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that there exist natural isomorphisms $\varepsilon : FG \Rightarrow \text{Id}$ and $\eta : \text{Id} \Rightarrow GF$. In this way $F(G(X)) \cong X$ instead of $F(G(X)) = X$.

It does not really matter here if we write $FG \Rightarrow \text{Id}$ or $\text{Id} \Rightarrow FG$ (the same for GF) because it is a natural isomorphism, but the above way of writing is to ensure consistency with an alternative definition of adjoint functors.

Remark 2.6.3. Let $X, Y \in \mathbf{Top}$ and $f : X \leftrightarrows Y : g$ be continuous maps. If $f \circ g \sim \text{id}$ and $g \circ f \sim \text{id}$ then X, Y are homotopy equivalent. Natural transformations are similar to the notion of homotopy.

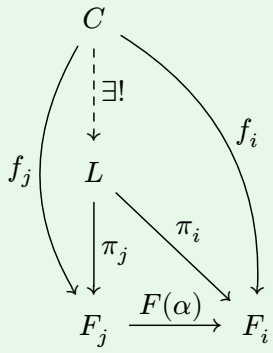
2.7. Limits and Colimits

Definition 2.7.1. Let I be a small category and $F : I \to \mathcal{A}$ be a functor. Then F is called a **diagram**. Denote $F(i) = F_i$ for all $i \in I$. A **cone** of F is an object C of $\mathcal{A}$ with morphisms $\{f_i : C \to F_i\}_{i \in I}$, such that for any $\alpha : j \to i$ in I,



commutes.

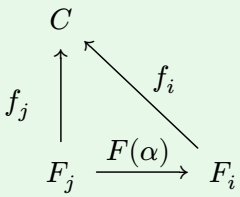
A limit is a universal cone; namely, L is a **limit** of F if it is a cone of F with $\{\pi_i : L \to F_i\}_{i\in I}$ and there exists a unique morphism $h : C \to L$ for any cone C of F with $\{f_i : C \to F_i\}_{i\in I}$ such that $f_i = \pi_i \circ h$ for all $i \in I$. We denote $L = \lim_I F$.



Notation 2.7.2. Sometimes we write $L = \lim F_i$ when I is clear from the context or is not important.

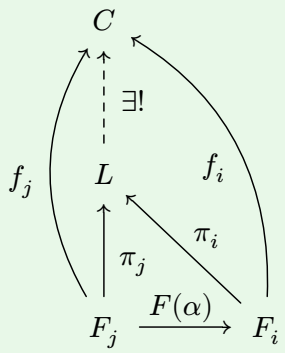
Dually, we define the colimit of F. This concept is important enough to be restated as follows.

Definition 2.7.3. Let I be a small category and $F : I \to \mathcal{A}$ be a diagram. Denote $F(i) = F_i$ for all $i \in I$. A **cocone** of F is an object C of $\mathcal{A}$ with morphisms $\{f_i : F_i \to C\}_{i\in I}$, such that for any $\alpha : j \to i$ in I,



commutes.

A colimit is a universal cocone; namely, L is a **colimit** of F if it is a cocone of F with $\{\pi_i : F_i \to L\}_{i\in I}$ and there exists a unique morphism $h : L \to C$ for any cocone C of F with $\{f_i : F_i \to C\}_{i\in I}$ such that $f_i = h \circ \pi_i$ for all $i \in I$. We denote $L = \operatorname{colim}_I F$.



Proposition 2.7.4. If any limit or colimit exists, then it is unique up to a unique isomorphism.

Notation 2.7.5. Hence we usually say "the" limit (or kernel, product, etc.) instead of "a" limit of a diagram.

Example 2.7.6. If I is a discrete category, then $\lim_I F = \prod_{i\in I} F_i$ is the product and $\operatorname{colim}_I F = \coprod_{i\in I} F_i$ is the coproduct.

Example 2.7.7. Let $I = \bullet \rightrightarrows \bullet$ be the category with two objects and two parallel morphisms between them. Let $F : I \to \mathcal{C}$ be a functor which maps I to

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

in $\mathcal{C}$. Then when $\lim_I F$ exists, we have two associated morphisms $h : \lim_I F \to A$ and $h' : \lim_I F \to A$, such that $f \circ h = h' = g \circ h$. We define the **equaliser** of f and g as this $h : \lim_I F \to A$, denoted as $\operatorname{eq}(f, g)$. We also dually define the **coequaliser** of f and g using $\operatorname{colim}_I F$, denoted as $\operatorname{coeq}(f, g)$, such that $\operatorname{coeq}(f, g) \circ f = \operatorname{coeq}(f, g) \circ g$. Continuing with Notation 2.2.9, we have

$$\operatorname{Eq}(f, g) \overset{\operatorname{eq}(f,g)}{\longrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\operatorname{coeq}(f,g)}{\longrightarrow} \operatorname{Coeq}(f, g).$$

Proposition 2.7.8. In a category with a zero object, $\mathrm{eq}(f, 0) = \ker f$ and $\mathrm{coeq}(f, 0) = \mathrm{coker}\, f$.

Proposition 2.7.9. An equaliser is a monomorphism. A coequaliser is an epimorphism.

Proposition 2.7.10. Let $L : \mathcal{A} \leftrightarrows \mathcal{B} : R$ be an adjunction and

$$L(\operatorname{colim} A_i) \cong \operatorname{colim} L(A_i)$$
$$R(\lim B_i) \cong \lim R(B_i)$$

Proof. Take $X \in \mathcal{B}$.

$$\begin{aligned}\mathrm{Hom}_{\mathcal{B}}(L(\operatorname{colim} A_i), X) &\cong \mathrm{Hom}_{\mathcal{A}}(\operatorname{colim} A_i, R(X)) \cong \lim \mathrm{Hom}_{\mathcal{A}}(A_i, R(X)) \\ &\cong \lim \mathrm{Hom}_{\mathcal{B}}(L(A_i), X) \cong \mathrm{Hom}_{\mathcal{B}}(\operatorname{colim} L(A_i), X).\end{aligned}$$

If we move colimit out of Hom, it becomes limit. (This has been seen for products and coproducts.) We then apply Yoneda Lemma to show $L(\operatorname{colim} A_i)$ and $\operatorname{colim} L(A_i)$ are isomorphic. ■

Remark 2.7.11. Left adjunction preserves colimits and right adjunction preserves limits. In particular, left adjunction preserves cokernels and are right exact; right adjunction preserves kernels and are left exact (to be defined later).

Proposition 2.7.12. A category $\mathcal{C}$ has all finite limits if and only if it has finite products and equalizers.

Proof. (Awodey 2010, Proposition 5.21). ■

2.8. Subobjects and Quotient Objects

(Awodey 2010, Section 5.1). This section offers some new vocabulary to describe things we already have seen.

Definition 2.8.1. Let A be an object of category $\mathcal{C}$. A **subobject** of A is a monomorphism $u : S \to A$.

Give two subobjects $u : S \to A$ and $v : T \to A$ of A, we define the relation of **inclusion** of subobjects by $u \subseteq v$ if and only if there exists $f : S \to T$ such that $u = v \circ f$. Such f is unique if it exists, since v is a monomorphism.

We say two subobjects $u : S \to A$ and $v : T \to A$ of A are **equivalent** if $u \subseteq v$ and $v \subseteq u$.

Proposition 2.8.2. Let $u : S \to A$ and $v : T \to A$ be two equivalent subobjects of A, then S and V are isomorphic objects.

Notation 2.8.3. Sometimes instead of saying $u : S \to A$ is a subobject of A, we may say S is a subobject of A when the monomorphism u is clear from the context.

Proposition 2.8.4. In category $\mathcal{C}$, $i : A \to B$ is the equaliser of $f, g : B \to C$ if and only if i is the largest subobject of B such that $f \circ i = g \circ i$. In particular, $i : A \to B$ is the kernel of $f : B \to C$ if and only if i is the largest subobject of B such that $f \circ i = 0$.

The dual concept of subobjects is **quotient objects**.

3. Abelian Categories

3.1. Ab-enriched Categories

We have seen, for example, that in $\mathbf{Vect}_k$ every hom-set not only is a collection (or set) of morphisms but also has some "additional structures", i.e., a vector space. This leads to the idea of enriched categories, where enriching means equipping the hom-sets with "additional structures". The following is an instance where every hom-set is an abelian group.

Definition 3.1.1. We call a category $\mathcal{C}$ **Ab-enriched** if every $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a abelian group, subject to bilinear morphism composition, namely

$$(f+g)\circ h = f\circ h + g\circ h \quad \text{and} \quad f\circ(k+h) = f\circ k + f\circ h$$

for all $f, g : Y \to Z$ and $h, k : X \to Y$.

Remark 3.1.2. An equivalent way to put the bilinearity is the following: the composition mappings

$$c_{XYZ} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z), \quad (f,g) \mapsto g\circ f$$

are group homomorphisms in each variable (Borceux 1994, vol. 2Definition 1.2.1).

Definition 3.1.3. Let $\mathcal{C}$ be an **Ab**-enriched category and $X, Y \in \mathcal{C}$. The **zero morphism** $0 \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is defined as the identity of the abelian group $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

However, note that an **Ab**-enriched category needs not have a zero object, so this is actually a redefinition of a zero morphism from Proposition 2.2.5. We will see later that the two definitions match when the zero object is present. Since group homomorphisms map identity to identity, we have the following:

Proposition 3.1.4. In an **Ab**-enriched category, let $X \xrightarrow{g} Y \xrightarrow{f} Z \xrightarrow{h} W$. If f is a zero morphism, then $f\circ g$ and $h\circ f$ are zero morphisms.

We can also define functors between **Ab**-enriched categories which respect the abelian group structures of the hom-set:

Definition 3.1.5. If $\mathcal{C}, \mathcal{D}$ are **Ab**-enriched, we call $F : \mathcal{C} \to \mathcal{D}$ an **additive functor** if

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is a group homomorphism for any $X, Y \in \mathcal{C}$.

Proposition 3.1.6. If $\mathcal{C}$ is an **Ab**-enriched category, then so is $\mathcal{C}^{\mathrm{op}}$.

Proof. The definition is self-dual. Namely, reversing all the arrows in $\mathcal{C}$ breaks neither the group structure on hom-sets nor the bilinear morphism composition. ■

An **Ab**-enriched category needs not have a zero object. Nevertheless, once it has an initial or final object, it has a zero object, as shown below.

Proposition 3.1.7. Let $*$ be an object in an **Ab**-enriched category, then the followings are equivalent:
(1) $*$ is a final object;
(2) $*$ is an initial object;
(3) $*$ is a zero object.

Proof. (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) is obvious. We only prove (1) $\Rightarrow$ (3), and (2) $\Rightarrow$ (3) follows from duality.

Suppose $*$ is a terminal object and let $\mathrm{id}_* : * \to *$ be the unique morphism in the abelian group of $\mathrm{Hom}_{\mathcal{C}}(*, *)$, and so $\mathrm{id}_* = 0$. For any object A and $f : * \to A$ (because $\mathrm{Hom}_{\mathcal{C}}(*, A)$ contains at least the zero morphism), we have

$$f = f \circ \mathrm{id}_* = f \circ 0 = 0 \in \mathrm{Hom}_{\mathcal{C}}(*, A).$$

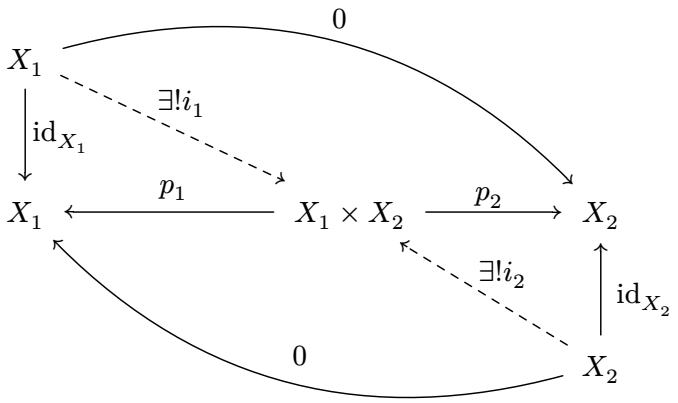
So there is a unique morphism from $*$ to A and therefore $*$ is also initial. ■

In fact, a final object is an empty product and an initial object an empty coproduct, and the previous result can be generalised.

Proposition 3.1.8. In an **Ab**-enriched category $\mathcal{C}$, let X_1, X_2 be two objects. Then
(1) If the product $X_1 \times X_2$ exists, then the coproduct $X_1 \sqcup X_2$ also exists and is isomorphic to $X_1 \times X_2$;
(2) If the coproduct $X_1 \sqcup X_2$ exists, then the product $X_1 \times X_2$ also exists and is isomorphic to $X_1 \sqcup X_2$.

Proof. (Monnet and Kremnitzer 2021, Proposition 3.7) and (Borceux 1994, vol. 2Proposition 1.2.4). We prove statement (1) and leave (2) to duality.

Suppose the product $X_1 \times X_2$ exists with projections $p_k : X_1 \times X_2 \to X_k$. By definition of products, there are unique morphisms $i_k : X_k \to X_1 \times X_2$ such that the following diagrams commute.



Explicitly, we have for $j, k \in \{1, 2\}$,

$$p_j \circ i_k = \begin{cases} \mathrm{id}_{X_j} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$p_1 \circ (i_1 p_1 + i_2 p_2) = p_1, \quad p_2 \circ (i_1 p_1 + i_2 p_2) = p_2.$$

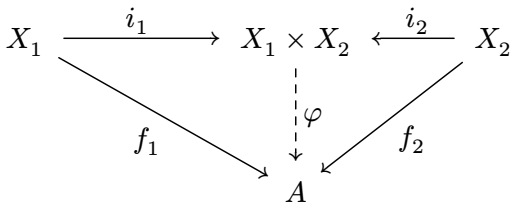
By definition of products, $\mathrm{id}_{X_1 \times X_2}$ is the unique morphism $h : X_1 \times X_2 \to X_1 \times X_2$ with $p_k \circ h = p_k$ for each k, so $i_1 p_1 + i_2 p_2 = \mathrm{id}_{X_1 \times X_2}$. We claim that

$$X_1 \overset{i_1}{\to} X_1 \times X_2 \overset{i_2}{\leftarrow} X_2$$

is a universal cocone and thus a coproduct. Suppose $X_1 \overset{f_1}{\to} A \overset{f_2}{\leftarrow} X_2$ is another cocone. Then we have a map

$$\varphi = f_1 \circ p_1 + f_2 \circ p_2 : X_1 \times X_2 \to A$$

such that for $k = 1, 2$, $\varphi \circ i_k = f_k$. This gives a commutative diagram



It remains to show that φ is unique. To see this, note that for any such φ we have

$$\begin{aligned} \varphi &= \varphi \circ \mathrm{id}_{X_1 \times X_2} \\ &= \varphi \circ (i_1 p_1 + i_2 p_2) \\ &= \varphi i_1 \circ p_1 + \varphi i_2 \circ p_2 \\ &= f_1 \circ p_1 + f_2 \circ p_2. \end{aligned}$$

■

Definition 3.1.9. Let $\mathcal{C}$ be an $\mathbf{Ab}$-enriched category and let $X_1, X_2 \in \mathcal{C}$. The **biproduct** of X_1 and X_2 is an object $X_1 \oplus X_2$ with morphisms $p_k : X_1 \oplus X_2 \to X_k$ and $i_k : X_k \to X_1 \oplus X_2$ for $k = 1, 2$, such that

- $p_k \circ i_k = 1_{X_k}$;
- $p_j \circ i_k = 0$ for $k \neq j$;
- $i_1 \circ p_1 + i_2 \circ p_2 = 1_{X_1 \oplus X_2}$.

Corollary 3.1.10. In an $\mathbf{Ab}$-enriched category, a binary biproduct is both a product and a coproduct, and a binary product (or a binary coproduct) is a biproduct.

Proof. This follows from the proof of Proposition 3.1.8. ■

Remark 3.1.11. This extends to all *finite* products and coproducts but does not extend to *infinite* products or coproducts.

Lemma 3.1.12. In an $\mathbf{Ab}$-enriched category, an additive functor preserves biproducts.

Proof. Notice that an additive functor preserves identity morphisms, zero morphisms, morphism compositions and morphism additions, and they are all we need in the definition of biproducts. ■

Being able to add and subtract parallel morphisms means we can rephrase the definitions for a monomorphism and epimorphism.

Proposition 3.1.13. In an $\mathbf{Ab}$-enriched category $\mathcal{C}$, $f : B \to C$ is a monomorphism if and only if $f \circ u = 0$ implies $u = 0$ for all $u : A \to B$. Dually, $f : B \to C$ is an epimorphism if and only if $v \circ f = 0$ implies $v = 0$ for all $v : C \to D$.

Proof. $f : B \to C$ is a monomorphism, if and only if $(f \circ -) : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ is injective for any A, if and only if $(f \circ -)$ (as a $\mathbb{Z}$-homomorphism) has kernel 0. ■

3.2. Additive Categories

Inspired by Proposition 3.1.7 and Proposition 3.1.8, we naturally define the following:

Definition 3.2.1. An $\mathbf{Ab}$-enriched category $\mathcal{C}$ is **additive** if it has all finite biproducts, including the zero object.

Now we can reconcile the two definitions we have had for zero morphisms.

Proposition 3.2.2. In an additive category $\mathcal{C}$, let $f : A \to B$. Then f is the identity of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ if and only if it can be factored as $A \to 0 \to B$.

Proof. Since $\mathrm{Hom}_{\mathcal{C}}(A, 0)$ has an unique element h, it must be the identity of the group. Similarly, $\mathrm{Hom}_{\mathcal{C}}(0, B)$ contains only the identity g. The composition $g \circ h$ is the identity of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ by Proposition 3.1.4. ■

Proposition 3.2.3. In an additive category, if a monomorphism $i : A \to B$ is a zero morphism, then A is the zero object. Dually, if an epimorphism $p : C \to D$ is a zero morphism, then D is the zero object.

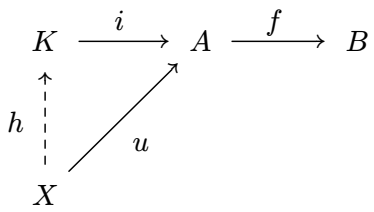
Proof. Take any X and $u : X \to A$, we have

$$X \overset{u}{\to} A \overset{i}{\to} B.$$

$i = 0$, so $i \circ u = 0$; but since i is monic, $u = 0$ by Proposition 3.1.13. Therefore there is a unique (zero) morphism from any X to A, so A is final and thus zero. ■

Proposition 3.2.4. (Rotman 2009, Proposition 5.89). Let $f : A \to B$ be a morphism in an additive category $\mathcal{C}$. If $\ker f$ exists, then f is monic if and only if $\ker f = 0$. Dually, if $\operatorname{coker} f$ exists, then f is epic if and only $\operatorname{coker} f = 0$.

Proof. Let $\ker f$ be $i : K \to A$. Suppose $i = 0$. Since we know a kernel is a monomorphism, by Proposition 3.2.3, $K = 0$. To show that f is monic, take any $u : X \to A$ such that $f \circ u = 0$. Then by the universal property of a kernel, there exists a unique morphism $h : X \to K$ such that $u = i \circ h$. Thus u factors through $K = 0$, so $u = 0$, proving f is monic by Proposition 3.1.13.



On the other hand, suppose f is monic. Then $\ker f = 0$ directly follows from Proposition 3.1.13. ■

3.3. Pre-abelian Categories

Now inspired by Proposition 3.2.4, we define the following:

Definition 3.3.1. An additive category $\mathcal{C}$ is **pre-abelian** if any morphism has a kernel and a cokernel.

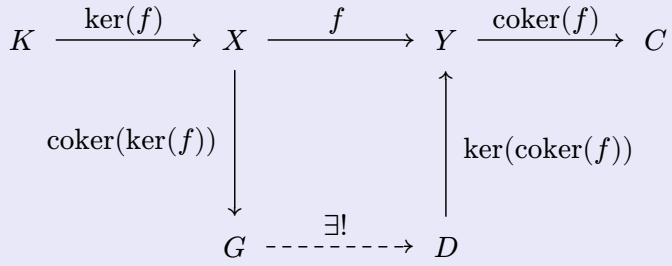
Corollary 3.3.2. Let f be a morphism in a pre-abelian category. f is monic if and only if $\ker f = 0$. Dually, f is epic if and only if $\operatorname{coker} f = 0$.

In fact, we get more than just kernels and cokernels:

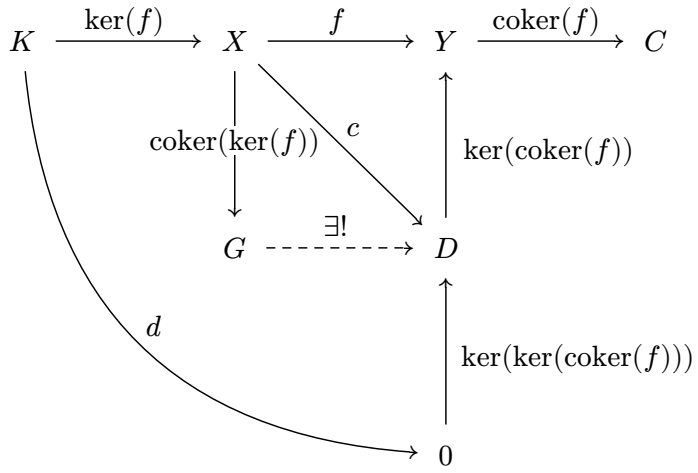
Proposition 3.3.3. A pre-abelian category has all finite limits and colimits.

Proof. Let $\mathcal{C}$ be a pre-abelian category. Since $\operatorname{eq}(f, g) = \ker(f - g)$, $\mathcal{C}$ has all equalisers and coequalisers. We also know that $\mathcal{C}$ has all finite products and coproducts as an additive category. Thus it has all finite limits and colimits by Proposition 2.7.12. ■

Proposition 3.3.4. If $\mathcal{C}$ is pre-abelian, for every morphism $f : X \to Y$, there exists a unique morphism $G \to D$ as shown below.



Proof. Since $\operatorname{coker}(f) \circ f = 0$, by the universal property of kernel, there exists $c : X \to D$ such that $f = \ker(\operatorname{coker}(f)) \circ c$. Since $f \circ \ker(f) = 0$, we have $\ker(\operatorname{coker}(f)) \circ c \circ \ker(f) = 0$. Now notice $\ker(\operatorname{coker}(f))$ is monic, and hence by Corollary 3.3.2, $\ker(\ker(\operatorname{coker}(f))) = 0$. By the universal property of kernel again, there exists $d : K \to 0$ such that $c \circ \ker(f) = \ker(\ker(\operatorname{coker}(f))) \circ d$. Thus $c \circ \ker(f)$ factors through the zero object and thus is 0. The desired morphism is obtained from the universal property of cokernel.



■

Definition 3.3.5. In a pre-abelian category, we define the **coimage** of a morphism f as

$$\operatorname{coim}(f) = \operatorname{coker}(\ker(f))$$

and **image** of f as

$$\operatorname{im}(f) = \ker(\operatorname{coker}(f)).$$

Continuing with Notation 2.2.9, we have $G = \operatorname{Coim}(f)$ and $D = \operatorname{Im}(f)$ in the above diagram. We call f **strict** if the map $\operatorname{Coim}(f) \to \operatorname{Im} f$ is an isomorphism.

3.4. Abelian Categories

Definition 3.4.1. A pre-ablian category is **abelian** if all morphisms are strict.

Corollary 3.4.2. In an abelian category, every morphism $f : X \to Y$ has a factorisation

$$X \overset{g}{\to} \mathrm{Im}(f) \overset{h}{\to} Y,$$

where g is an epimorphism and h is a monomorphism.

Proof. Notice $g = \mathrm{coker}(\ker(f)) = \mathrm{coim}(f)$ and $h = \ker(\mathrm{coker}(f)) = \mathrm{im}(f)$. ∎

We can always write $f = \mathrm{im}(f) \circ \mathrm{coim}(f)$ and consider $\mathrm{im}(f)$ as a subobject of Y.

Remark 3.4.3. The followings are two equivalent definitions of an abelian category:

- A pre-abelian category where every monomorphism is a kernel and every epimorphism is a cokernel;
- A pre-abelian category where every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

We prove part of the equivalence:

Proposition 3.4.4. In an abelian category, every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Proof. Use the diagram in the proof of Proposition 3.3.4. Let f be a monomorphism, then $\ker(f) = 0$ and $K = 0$. It is not to hard to find $G = X$ and $\mathrm{coker}(\ker(f)) = \mathrm{id}_X$. Since D and G are isomorphic, we see that X is isomorphic to D and thus $f = \ker(\mathrm{coker}(f))$. ∎

Remark 3.4.5. Now it is time to give a list of properties that abelian categories have, packing everything we have picked up along the way:

- Every hom-set is an abelian group subject to bilinear morphism composition;
- It has a zero object and has a zero morphism between any two objects, which is the identity of the abelian group and factors through 0;
- It has all limits and colimits;
- Any finite product and coproduct coincide as the biproduct;
- f is monic if and only if $f \circ u = 0$ implies $u = 0$, if and only if $\ker f = 0$, if and only if $f = \mathrm{im}(f)$;
- g is epic if and only if $v \circ g = 0$ implies $v = 0$, if and only if $\mathrm{coker}\, g = 0$, if and only if $g = \mathrm{colim}(g)$;
- f is monic and $f = 0$ implies the domain of f is 0;
- g is epic and $g = 0$ implies the codomain of g is 0;
- $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism;
- Any f can be factorised as $f = \ker(\mathrm{coker}(f)) \circ \mathrm{coker}(\ker(f)) = \mathrm{im}(f) \circ \mathrm{coim}(f)$.

We now introduce the most important member in the family of abelian categories.

Proposition 3.4.6. For any ring R, the category R-**Mod** is an abelian category. In particular, **Ab** is an abelian category.

Proof. (R-**Mod** is **Ab**-enriched.) For any $A, B \in R$-**Mod**, the set $\mathrm{Hom}_R(A, B)$ of module homomorphisms $A \to B$ can be naturally seen as an abelian group under pointwise addition. It is easy to check that the composition is bilinear.

(R-**Mod** is additive.) We know that the direct sum exists as a coproduct for any finite family of modules $(M_i)_{i\in I}$ in R-**Mod**.

(R-**Mod** is pre-abelian.) Let $f : A \to B$ be a morphism in R-**Mod**. Then

$$\mathrm{Ker}(f) = \{a \in A : f(a) = 0\}$$

with $\ker(f) : \mathrm{Ker}(f) \to A$ being the inclusion map, is a categorical kernel. Also,

$$\mathrm{Coker}(f) = B/\operatorname{Im}(f)$$

where $\operatorname{Im}(f) = \{f(a) \in B : a \in A\}$, with $\mathrm{coker}(f) : B \to \mathrm{Coker}(f)$ being the quotient map, is a categorical cokernel.

(R-**Mod** is abelian.) We find

$$\mathrm{Coker}(\ker(f)) = A/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$$

by the First Isomorphism Theorem and

$$\mathrm{Ker}(\mathrm{coker}(f)) = \operatorname{Im}(f)$$

by construction. Hence the image and coimage coincide up to isomorphism, i.e., any f is strict. ∎

Remark 3.4.7. Note that the product and coproduct of a family $(M_i)_{i\in I}$ coincide when I is finite but differ when I is infinite:

$$\bigsqcup_{i\in I} M_i = \bigoplus_{i\in I} M_i = \left\{(m_i)_{i\in I} \mid m_i \in M_i, m_i = 0 \text{ for almost all } i\right\},$$

$$\prod_{i\in I} M_i = \left\{(m_i)_{i\in I} \mid m_i \in M_i\right\}.$$

Proposition 3.4.8. In R-**Mod**, a monomorphism is equivalent to an injective homomorphism and an epimorphism is equivalent to a surjective homomorphism.

Example 3.4.9. If $\mathcal{A}$ is an abelian category and $\mathcal{C}$ is any small category, then the category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ is abelian.

Example 3.4.10. The category of Banach spaces over $\mathbb{R}$ is not an abelian category, but a **quasi-abelian category**.

3.5. Exact Sequences and Functors

Note 3.5.1. All discussions in this section are limited to an abelian category.

We have trekked a long way to establish abelian categories. The key element that we seek from an abelian category is the notion of exactness:

Definition 3.5.2. In an abelian category, a sequence of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is called **exact** at B if $\ker g = \operatorname{im} f$ (as equivalent subobjects of B).

Definition 3.5.3. In an abelian category, a **short exact sequence** $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact at A, B and C, or "exact everywhere".

Lemma 3.5.4. $\operatorname{im}(0 \to A) = 0$ and $\operatorname{im}(A \to 0) = 0$.

Proposition 3.5.5. $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence if and only if f is monic, g is epic, and $\ker g = \operatorname{im} f$.

Proof.
- Exactness at $A \Leftrightarrow \ker f = \operatorname{im}(0 \to A) = 0 \Leftrightarrow f$ is monic.
- Exactness at $B \Leftrightarrow \ker g = \operatorname{im} f$.
- Exactness at $C \Leftrightarrow \operatorname{im} g = \ker(C \to 0) = \mathrm{id}_C \Leftrightarrow g = \operatorname{coim}(g) \Leftrightarrow g$ is epic.

■

Proposition 3.5.6. If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence, then $f = \ker g$ and $g = \operatorname{coker} f$.

Proof. f is monic, so $f = \operatorname{im}(f) = \ker(g)$. g is epic, so $g = \operatorname{coim}(g) = \operatorname{coker}(\ker(g)) = \operatorname{coker}(f)$. ■

Corollary 3.5.7. $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ can be rewritten as

$$0 \to \operatorname{Im}(f) \to B \xrightarrow{\operatorname{coker}(f)} \operatorname{Coker}(f) \to 0$$

or

$$0 \to \mathrm{Ker}(g) \overset{\ker(g)}{\to} B \to \mathrm{Coim}(g) \to 0.$$

Proposition 3.5.8. If $A \overset{f}{\to} B \to C \to D \overset{g}{\to} E$ is an exact sequence, then

$$0 \to \mathrm{Coker}(f) \to C \to \mathrm{Ker}(g) \to 0$$

is a short exact sequence.

Definition 3.5.9. A short exact sequence $0 \to A \to B \to C \to 0$ is **split** if B is isomorphic to $A \oplus C$.

Lemma 3.5.10 (Splitting Lemma). Let $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ be a short exact sequence. The followings are equivalent:
(1) The short exact sequence is split;
(2) There exists a **retraction**[2] $r : B \to A$ such that $r \circ f = \mathrm{id}_A$;
(3) There exists a **section** $s : C \to B$ such that $g \circ s = \mathrm{id}_C$.

Proof. Although it is possible to give a purely category-theoretic proof, as can be seen (Noix07 2014), we give a proof in R-**Mod**, which is in fact sufficient in view of Theorem 3.7.11.

(1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial by the definition of biproducts.

(2) $\Rightarrow$ (1). We first claim that $B = \mathrm{Im}\, f + \mathrm{Ker}\, r$. Take any $b \in B$, then plainly $b = fr(b) + (b - fr(b))$. Since $r(b - fr(b)) = r(b) - rfr(b) = 0$, we have $b - fr(b) \in \mathrm{Ker}\, r$. Also obviously $fr(b) \in \mathrm{Im}\, f$.

We further claim that $B = \mathrm{Im}\, f \oplus \mathrm{Ker}\, r$. Suppose $b \in \mathrm{Im}\, f \cap \mathrm{Ker}\, r$, then there exists $a \in A$ such that $b = f(a)$; also $r(b) = 0$. Then $0 = rf(a) = a$, so $b = f(a) = 0$.

Now we claim that $\mathrm{Ker}\, r \cong C$; in particular, the restriction $g|_{\mathrm{Ker}\, r} : \mathrm{Ker}\, r \to C$ is an isomorphism. Take any $c \in C$, then since g is a surjection, there exists some $f(a) + k \in B$, where $a \in A$ and $k \in \mathrm{Ker}\, r$, such that $g(f(a) + k) = c$. Note that $gf(a) = 0$, because $f(a) \in \mathrm{Im}\, f = \mathrm{Ker}\, g$ by exactness at B, so for any $c \in C$, there exists $k \in \mathrm{Ker}\, r$ such that $g(k) = c$. Thus $g|_{\mathrm{Ker}\, r}$ is surjective. On the other hand, if $g(k) = 0$ for $k \in \mathrm{Ker}\, r$, then $k \in \mathrm{Ker}\, g = \mathrm{Im}\, f$, but $\mathrm{Im}\, f \cap \mathrm{Ker}\, r = \{0\}$, so $k = 0$. Thus $g|_{\mathrm{Ker}\, r}$ is injective.

Finally, observe that f is an injection, so $\mathrm{Im}(f) \cong A$.

(3) $\Rightarrow$ (1). The proof is similar as above and thus omitted. ∎

[2]The terms "retraction" and "section" come from algebraic topology, but for our purpose they are nothing more than certain morphisms.

Corollary 3.5.11. Let M, S, T be R-modules.
- If $M = S \oplus T$ and $S \subseteq N \subseteq M$, then $N = S \oplus (N \cap T)$.
- If $M = S \oplus T$ and $S' \subseteq S$, then $M/S' = S/S' \oplus (T + S')/S'$.

Proof. (Rotman 2009, Corollary 2.24). ∎

Definition 3.5.12. An additive functor $F : \mathcal{C} \to \mathcal{D}$ is called
- **right exact** if the exactness of $A \to B \to C \to 0$ implies the exactness of $F(A) \to F(B) \to F(C) \to 0$;
- **left exact** if the exactness of $0 \to A \to B \to C$ implies the exactness of $0 \to F(A) \to F(B) \to F(C)$;
- **exact** if the exactness of $0 \to A \to B \to C \to 0$ implies the exactness of $0 \to F(A) \to F(B) \to F(C) \to 0$,

for any $A, B, C \in \mathcal{C}$.

Remark 3.5.13. By definition, *right exactness preserves cokernels*, since C is the cokernel of the map $A \to B$ and $F(C)$ is the cokernel of the map $F(A) \to F(B)$. Similarly, *left exactness preserves kernels*.

Lemma 3.5.14. Let $\mathcal{A}$ be an abelian category. Let $M \in \mathcal{A}$. The functor

$$\mathrm{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \to \mathbf{Ab}$$

is left exact.

Proof. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ be exact in $\mathcal{A}$, then we want to prove

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M, A) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathcal{A}}(M, B) \xrightarrow{g\circ -} \mathrm{Hom}_{\mathcal{A}}(M, C)$$

is exact in $\mathbf{Ab}$.

Exactness at $\mathrm{Hom}_{\mathcal{A}}(M, A)$ is equivalent to $(f \circ -)$ being monic, so let us calculate $\mathrm{Ker}(f \circ -)$. Let $u \in \mathrm{Hom}_{\mathcal{A}}(M, A)$ such that $(f \circ -)(u) = 0$, i.e., $f \circ u = 0$. But f is monic, so $u = 0$, and thus $\mathrm{Ker}(f \circ -) = 0$ and $(f \circ -)$ is monic.

Exactness at $\mathrm{Hom}_{\mathcal{A}}(M, B)$ is equivalent to $\mathrm{Ker}(g \circ -) = \mathrm{Im}(f \circ -)$. To show that $\mathrm{Ker}(g \circ -) \subseteq \mathrm{Im}(f \circ -)$, let $v \in \mathrm{Ker}(g \circ -)$. Then $v : M \to B$ such that $g \circ v = 0$. Note that $A = \mathrm{Ker}(g)$ and $f = \ker(g)$, so by the universal property of kernel, there exists $h : M \to A$ such that $v = f \circ h$, hence $v \in \mathrm{Im}(f \circ -)$. On the other hand, to show that $\mathrm{Im}(f \circ -) \subseteq \mathrm{Ker}(g \circ -)$, notice that if $v \in \mathrm{Im}(f \circ -)$, then $v = f \circ h$ for some h and then $g \circ v = g \circ f \circ h = 0$ since $g \circ f = 0$. ∎

Remark 3.5.15. The functor $\mathrm{Hom}_{\mathcal{A}}(M, -)$ fails to be exact in general because it does not necessarily send an epimorphism to an epimorphism. For a counterexample, let $\mathcal{A} = \mathbf{Ab}$ (where an epimorphism is equivalent to a surjective homomorphism) and $M = \mathbb{Z}/2\mathbb{Z}$. The quotient map $h : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ is an surjective homomorphism. On the other hand, for any abelian group A, an element in $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, A)$ (i.e., a group homomorphism $\mathbb{Z}/2\mathbb{Z} \to A$) is uniquely determined

by an element in A with order 2. Hence $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$ and $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, and we see the induced map

$$(h \circ -) : \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$$

cannot be surjective.

Corollary 3.5.16. Dually, $\mathrm{Hom}_{\mathcal{A}}(-, M) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is also left exact.

Note 3.5.17. What does left exactness mean for a contravariant functor? If $X \to Y \to Z \to 0$ is exact in $\mathcal{A}$, then $0 \to Z \to Y \to X$ is exact in $\mathcal{A}^{\mathrm{op}}$, and $0 \to \mathrm{Hom}_{\mathcal{A}}(Z, M) \to \mathrm{Hom}_{\mathcal{A}}(Y, M) \to \mathrm{Hom}_{\mathcal{A}}(X, M)$ is exact in $\mathbf{Ab}$.

3.6. Projective and Injective Objects

Definition 3.6.1. Let $\mathcal{A}$ be an abelian category. An object P is called **projective** if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact. Dually, an object I is called **injective** if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact.

In other words, P is projective if for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$,

$$0 \to \mathrm{Hom}_{\mathcal{A}}(P, X) \to \mathrm{Hom}_{\mathcal{A}}(P, Y) \to \mathrm{Hom}_{\mathcal{A}}(P, Z) \to 0$$

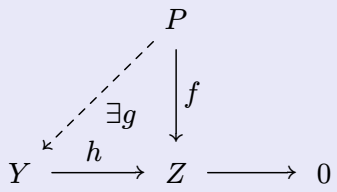
is a short exact sequence.

Proposition 3.6.2. The followings are equivalent:

(1) P is a projective object;

(2) For any epimorphism $h : Y \to Z$, the induced map $(h \circ -) : \mathrm{Hom}_{\mathcal{A}}(P, Y) \to \mathrm{Hom}_{\mathcal{A}}(P, Z)$ is surjective;

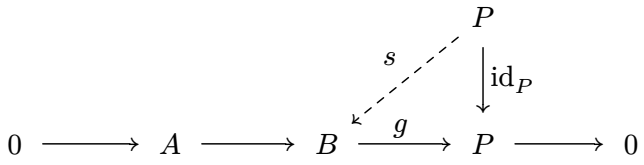
(3) For any epimorphism $h : Y \to Z$ and any morphism $f : P \to Z$, there exists (not necessarily unique) $g : P \to Y$ such that $f = h \circ g$, i.e. the following commutes (which we refer to as the **lifting property**):



(4) Any short exact sequence $0 \to A \to B \to P \to 0$ splits.

Proof. (1) $\Rightarrow$ (2) is obvious; (2) $\Rightarrow$ (1) by Lemma 3.5.14. (2) $\Leftrightarrow$ (3) is also obvious.

(3) $\Rightarrow$ (4).



Since $g : B \to P$ is an epimorphism, we can always find $s : P \to B$ such that $g \circ s = \mathrm{id}_P$ by the lifting property. Then (4) holds by Splitting Lemma 3.5.10.

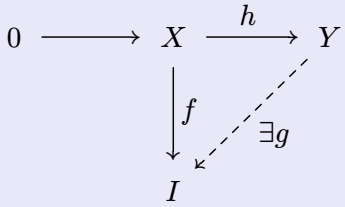
(4) $\Rightarrow$ (3). See (Rafael 2019). ■

Corollary 3.6.3. Dually, the followings are equivalent:

(1) I is injective;

(2) For any monomorphism $h : X \to Y$, the induced map $(- \circ h) : \mathrm{Hom}_{\mathcal{A}}(Y, I) \to \mathrm{Hom}_{\mathcal{A}}(X, I)$ is surjective;

(3) For any monomorphism $h : X \to Y$ and any $f : X \to I$, there exists $g : Y \to I$ such that $f = g \circ h$, i.e., the following commutes (which we refer to as the **extension property**):



(4) Any short exact sequence $0 \to I \to A \to B \to 0$ splits.

3.7. Categories of Modules

Proposition 3.7.1. Ring R viewed as an object in R-**Mod** is projective.

Proof. It is equivalent to say the functor $\mathrm{Hom}_R(R, -)$ is exact. In fact, $\mathrm{Hom}_R(R, M) \cong M$ because any module morphism $\varphi : R \to M$ is entirely determined by $\varphi(1_R)$. Given any short exact sequence $0 \to M \to M' \to M'' \to 0$, if we apply $\mathrm{Hom}_R(R, -)$, we get the same short exact sequence, which is exact. ■

Note 3.7.2. In R-**Mod**, we have

$$\mathrm{Hom}_R\left(R, \bigoplus_{i \in I} M_i\right) = \bigoplus_{i \in I} M_i = \bigoplus_{i \in I} \mathrm{Hom}_R(R, M_i).$$

This does not follow from the universal property of the direct sum; this is because R is special.

Definition 3.7.3. Let $\mathcal{A}$ be an additive category. We call an object C **compact** if the canonical morphism

$$\coprod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(C, G_i) \to \mathrm{Hom}_{\mathcal{A}}\left(C, \coprod_{i\in I} G_i\right)$$

is an isomorphism for any family $\{G_i\}_{i\in I}$ of objects in $\mathcal{A}$ such that $\coprod_{i\in I} G_i$ exists.

Remark 3.7.4. You might find different definitions for an arbitrary category (not necessarily additive), but they are equivalent under the additive context.

Definition 3.7.5. In a category $\mathcal{C}$ with coproducts, an object G is called a **generator** if for any $X \in \mathcal{C}$, there is an epimorphism $\coprod_I G \to X \to 0$.

Lemma 3.7.6. R is a generator of R-**Mod**.

Proof. Recall Corollary 1.9. ∎

Lemma 3.7.7. In an abelian category $\mathcal{A}$, any hom-set $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ can be seen as a right module over ring $\mathrm{End}_{\mathcal{A}}(X)$, or equivalently a left module over $\mathrm{End}_{\mathcal{A}}(X)^{\mathrm{op}}$.

Proof. First notice $\mathrm{End}_{\mathcal{A}}(X)$ is indeed a ring with composition as multiplication. Take any $m \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $r \in \mathrm{End}_{\mathcal{A}}(X)$. Define the multiplication mr as $m \circ r \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$. It is easy to verify that this makes $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ a right module over $\mathrm{End}_{\mathcal{A}}(X)$. ∎

Theorem 3.7.8 (Morita's Theorem). Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has (small) coproducts. Assume that P is a compact, projective generator. Let ring $R = \mathrm{End}_{\mathcal{A}}(P)$, then the functor

$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{Mod}\text{-}R$$

is an equivalence of categories.

Note 3.7.9. If $\mathcal{A} = S$-**Mod** for some ring S, we have observed that S (as an object of S-**Mod**) is a compact, projective generator. In this case, $R = \mathrm{End}_S(S)$. We observe that any module homomorphism $\varphi : S \to S$ is uniquely determined by $\varphi(1) \in S$ with $\varphi(s) = s\varphi(1)$, and the composition of two homomorphisms $\varphi_1, \varphi_2 : S \to S$ is in the opposite direction of multiplication in S:

$$\varphi_1(\varphi_2(s)) = s\varphi_2(1)\varphi_1(1)$$

Therefore, $R = \mathrm{End}_S(S) = S^{\mathrm{op}}$. Thus, indeed, we have S-**Mod** is equivalent to **Mod**-R, which is **Mod**-S^{op}.

Proof. (Rotman 2009, Theorem 5.55) and (Pareigis 1970, p. 211). Denote $F := \mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to$ **Mod**-R. Using the definition of categorical equivalence, we want to construct another functor $G :$ **Mod**-$R \to \mathcal{A}$ and show FG and GF are naturally isomorphic to identity functors. We see that in this way G should be left adjoint to F, so G must preserves colimits and in particular be right exact.

Inspired by the discussion above, we define G in the following way. We first set $G(R) = P$ and $G(R^{\oplus I}) = P^{\oplus I}$. Any morphism $f : R^{\oplus J} \to R^{\oplus I}$ can be represented by a (possibly infinite) matrix with entries $a_{ij} \in R$ for all $i \in I$ and $j \in J$. However, notice that $R = \mathrm{End}_{\mathcal{A}}(P)$ by definition and thus the same matrix $\left(a_{ij}\right)_{i\in I, j\in J}$ can also be seen as a morphism $P^{\oplus J} \to P^{\oplus I}$, which is defined to be $G(f)$. Now, for any R-module M, we can find a presentation

$$R^{\oplus J} \xrightarrow{f} R^{\oplus I} \to M \to 0$$

Under G, this becomes

$$P^{\oplus J} \xrightarrow{G(f)} P^{\oplus I} \to G(M) \to 0$$

where we define $G(M) = \mathrm{Coker}(G(f))$. It can be verified that G is a functor.

Since P is a projective object, F is exact and preserves cokernels; since P is compact, F preserves direct sums. On the other hand, G is right exact and preserves direct sums by construction. Hence the composites FG and GF are right exact and preserves direct sums. Now we check FG and GF are naturally isomorphic to identity functors.

For $FG :$ **Mod**-$R \to$ **Mod**-R, we have

$$FG(R) = F(P) = \mathrm{Hom}_{\mathcal{A}}(P, P) = R$$

and hence $FG(R^{\oplus I}) = R^{\oplus I}$. Now for any $M \in$ **Mod**-R, there is a commutative diagram

$$\begin{array}{ccccccc} R^{\oplus J} & \longrightarrow & R^{\oplus I} & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ FG(R^{\oplus J}) & \longrightarrow & FG(R^{\oplus I}) & \longrightarrow & FG(M) & \longrightarrow & 0 \end{array}$$

Since FG preserves cokernels, we see that $FG(M) \cong M$. Hence FG is naturally isomorphic to the identity functor of **Mod**-R.

For $GF : \mathcal{A} \to \mathcal{A}$, we have $GF(P) = G(R) = P$, so $GF(P^{\oplus I}) = P^{\oplus I}$. Now take any $X \in \mathcal{A}$, since P is a generator, we can find

$$P^{\oplus J} \to P^{\oplus I} \to X \to 0$$

A similar argument as before gives the result. ∎

Remark 3.7.10. $\mathcal{A}$ can have more than one compact, projective generator, say P_1 and P_2. Then $A = \mathrm{End}_{\mathcal{A}}(P_1)^{\mathrm{op}}$- **Mod** $= \mathrm{End}_{\mathcal{A}}(P_2)^{\mathrm{op}}$- **Mod**, where rings $\mathrm{End}_{\mathcal{A}}(P_1)$ and $\mathrm{End}_{\mathcal{A}}(P_2)$ are not necessarily isomorphic. This is **Morita equivalence** of rings.

For example, consider $\mathbf{Vect}_k$ for some field k. Then k and k^n are both compact, projective generators of $\mathbf{Vect}_k$. Then k and $M_n(k)$ ($n \times n$ matrices over k) both are equivalent to $\mathbf{Vect}_k$ as categories.

Theorem 3.7.11 (Freyd-Mitchell Embedding Theorem). If $\mathcal{A}$ is a small abelian category, there is a ring R and an exact, fully faithful embedding functor $\mathcal{A} \to R\text{-}\mathbf{Mod}$.

Proof. (Weibel 1994, p. 25). ∎

This theorem indicates that we can embed an abstract category into a concrete one. From a practical perspective, we can prove any reasonable statements for R-$\mathbf{Mod}$ and they will also hold for abelian categories. An example is the following.

Lemma 3.7.12 (Snake Lemma). Suppose we have a commutative diagram of objects in an abelian category or R-$\mathbf{Mod}$

$$\begin{array}{ccccccccc} & & A' & \xrightarrow{i'} & B' & \xrightarrow{p'} & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & & \end{array}$$

such that the rows are exact, then there is an exact sequence

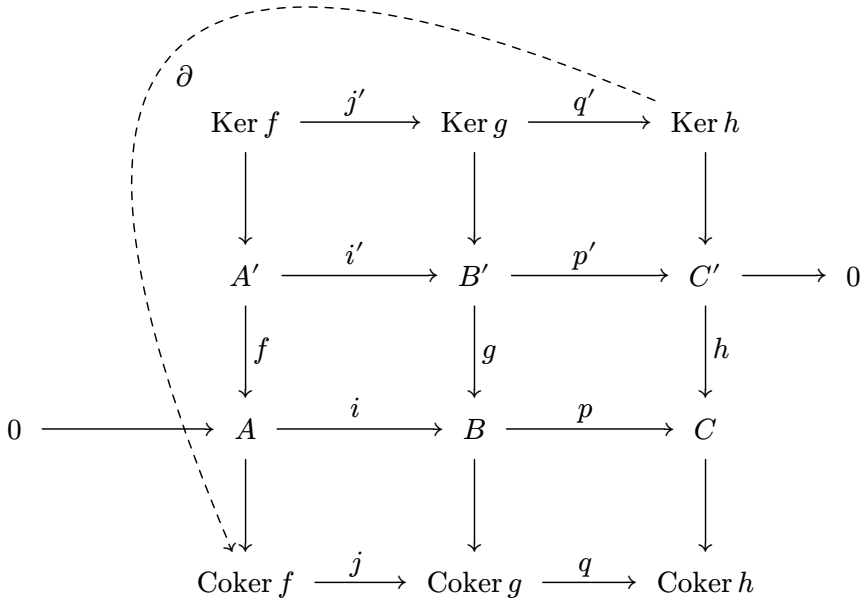
$$\operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h \xrightarrow{\partial} \operatorname{Coker} f \to \operatorname{Coker} g \to \operatorname{Coker} h$$

where the **connecting (homo)morphism** ∂ is given by a well-defined formula

$$\partial(c') = i^{-1} g {p'}^{-1}(c') + \operatorname{Im}(f)$$

where ${p'}^{-1}$ means finding some element $b' \in B'$ such that $p'(b') = c'$ and so on. Further, if $A' \to B'$ is monic, so is $\operatorname{Ker} f \to \operatorname{Ker} g$. If $B \to C$ is epic, so is $\operatorname{Coker} g \to \operatorname{Coker} h$.

Proof. A detailed proof can be seen (Gardner 2023). We have the following commutative diagram:



In the first row, consider map $j' := i'|_{\operatorname{Ker} f} : \operatorname{Ker} f \to B'$. We claim that $j' : \operatorname{Ker} f \to \operatorname{Ker} g$. Indeed, take any $a' \in \operatorname{Ker} f \subseteq A'$, we have

$$g(j'(a')) = g(i'(a')) = i(f(a')) = i(0) = 0.$$

Then $j'(a') \in \operatorname{Ker} g$ and thus $j' : \operatorname{Ker} f \to \operatorname{Ker} g$. Similarly, $q' := p'|_{\operatorname{Ker} g} : \operatorname{Ker} g \to \operatorname{Ker} h$. We then see the first row is exact because of the exactness of $A' \to B' \to C'$. Also, if i' is an injection, i.e., $\operatorname{Ker}(i') = 0$, then obviously $\operatorname{Ker}(j') = 0$.

In the last row, define $j : \operatorname{Coker}(f) \to \operatorname{Coker}(g)$ as $a + \operatorname{Im}(f) \mapsto i(a) + \operatorname{Im}(g)$ for any $a \in A$. We claim that this map is well-defined. If $a_1, a_2 \in A$ such that $a_1 + \operatorname{Im}(f) = a_2 + \operatorname{Im}(f)$, then $a_1 - a_2 \in \operatorname{Im}(f)$, thus there exists $a' \in A'$ so that $a_1 - a_2 = f(a')$. Then $i(a_1 - a_2) = i(f(a')) = g(i'(a')) \in \operatorname{Im}(g)$. Then

$$j(a_1 + \operatorname{Im}(f)) = i(a_1) + \operatorname{Im}(g) = i(a_2) + \operatorname{Im}(g) = j(a_2 + \operatorname{Im}(f)).$$

So j is well-defined. Similarly, we can define $q : \operatorname{Coker} g \to \operatorname{Coker} h$ and show the exactness of the last row. We can also see that the surjection of p implies the surjection of q.

Now all arrows except ∂ are clear. Pick any $c' \in \operatorname{Ker} h \subseteq C'$. Since p' is surjective, there exists $b' \in B'$ so that $p'(b') = c'$. Now $0 = h(c') = h(p'(b')) = p(g(b'))$, so $g(b') \in \operatorname{Ker} p = \operatorname{Im} i$, and there exists unique $a \in A$ such that $i(a) = g(b')$. We thus define $\partial : \operatorname{Ker} h \to \operatorname{Coker} f$ as $\partial(c') = a + \operatorname{Im}(f)$. We claim this is a well-defined function. Then it suffices to show for any two choices b'_1, b'_2 of b' and corresponding choices a_1, a_2 of a, $\partial(c')$ gives the same value. Since $p'(b'_1) = p'(b'_2) = c'$, we have $b'_1 - b'_2 \in \operatorname{Ker}(p') = \operatorname{Im}(i')$. Thus we can write $b'_1 - b'_2 = i'(a')$ for some $a' \in A'$. Then $i(a_1 - a_2) = g(b'_1 - b'_2) = g(i'(a')) = i(f(a'))$, but i is injective, and hence $a_1 - a_2 = f(a') \in \operatorname{Im} f$.

We omit the proof of the exactness at $\operatorname{Ker} h$ and $\operatorname{Coker} f$. ■

4. Tensor Product of Modules

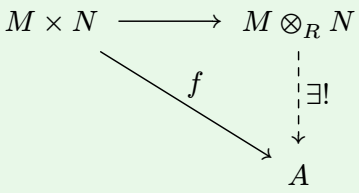
4.1. Existence and Functoriality

Definition 4.1.1. Let R be a ring. Consider right R-module M, left R-module N and abelian group A. A map $f : M \times N \to A$ is called a **balanced product** (or we say it is R-**biadditive**) if it satisfies:

$$\begin{aligned} f(x + x', y) &= f(x, y) + f(x', y), \\ f(x, y + y') &= f(x, y) + f(x, y'), \\ f(xr, y) &= f(x, ry). \end{aligned}$$

for all $x, x' \in M$, $y, y' \in N$ and $r \in R$.

Definition 4.1.2. The **tensor product** of a right R-module M and a left R-module N is an abelian group $M \otimes_R N$ with a balance product $M \times N \to M \otimes_R N$ such that for any balanced product $f : M \times N \to A$, there exists a unique group homomorphism $M \otimes_R N \to A$ that makes the diagram commutes:



We might simply say $M \otimes_R N$ is the tensor product of M and N.

Remark 4.1.3. In other words, $M \otimes_R N$ is an initial object of the category of all balanced products $M \times N \to A$ (where a morphism is a group homomorphism $A \to A'$).

Lemma 4.1.4. The tensor product $M \times N \to M \otimes_R N$ exists, up to a unique isomorphism, for any right R-module M and left R-module N.

Proof. Consider the free abelian group F with basis $M \times N$, and let $i : M \times N \to F$ be the inclusion map. F has a subgroup I generated by all elements of the following forms

$$\begin{gathered} (x + x', y) - (x, y) - (x', y) \\ (x, y + y') - (x, y) - (x, y') \\ (xr, y) - (x, ry) \end{gathered}$$

for $x, x' \in M$, $y, y' \in N$ and $r \in R$. Denote $M \otimes_R N := F/I$, and denote the coset $(x, y) + I$ by $x \otimes y$, and define

$$h : M \times N \to M \otimes_R N$$
$$(x, y) \mapsto x \otimes y$$

It is clear that h is biadditive.

Let $f : M \times N \to A$ be a balanced product, then there exists a homomorphism $f' : F \to A$ which linearly extends f such that $f = f' \circ i$. Now we see that $I \subseteq \operatorname{Ker} f'$ and thus f' induces a map

$$\hat{f} : F/I \to A$$
$$(x, y) + I \mapsto f'(x, y) = f(x, y)$$

which is the same as saying

$$\hat{f} : M \otimes_R N \to A$$
$$x \otimes y \mapsto f(x, y)$$

Thus we see that $\hat{f} \circ h = f$, and we can conclude that $M \otimes_R N$ is a tensor product of M and N. The uniqueness follows from the universal property. ■

Lemma 4.1.5. Let $\varphi : M \to M'$ and $\psi : N \to N'$ be module homomorphisms, then there exists a unique group homomorphism $\varphi \otimes \psi : M \otimes_R N \to M' \otimes_R N'$, such that

$$\varphi \otimes \psi : m \otimes n \mapsto \varphi(m) \otimes \psi(n)$$

Proof. The function

$$f : M \times N \to M' \otimes_R N'$$
$$(m, n) \mapsto \varphi(m) \otimes \psi(n)$$

is R-biadditive. Therefore, f induces a unique homomorphism

$$\hat{f} : M \otimes_R N \to M' \otimes_R N'$$
$$m \otimes n \mapsto \varphi(m) \otimes \psi(n)$$

which we write as $\varphi \otimes \psi$. ■

Corollary 4.1.6. $(\varphi' \circ \varphi) \otimes (\psi' \circ \psi) = (\varphi' \otimes \psi') \circ (\varphi \otimes \psi)$ for any $M \overset{\varphi}{\to} M' \overset{\varphi'}{\to} M''$ and $N \overset{\psi}{\to} N' \overset{\psi'}{\to} N''$.

Proof. Both send $m \otimes n$ to $\varphi'(\varphi(m)) \otimes \psi'(\psi(n))$, but such a homomorphism should be unique. ■

Corollary 4.1.7. Let M be a right R-module and N be a left R-module, then we have functors

$$M \otimes_R - : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$$
$$B \mapsto M \otimes_R B$$
$$(g : B \to B') \mapsto \mathrm{id}_M \otimes g$$

$$
\begin{aligned}
- \otimes_R N : \mathbf{Mod}\text{-}R &\to \mathbf{Ab} \\
A &\mapsto A \otimes_R N \\
(f : A \to A') &\mapsto f \otimes \mathrm{id}_N
\end{aligned}
$$

4.2. Bimodules and Bilinearity

Definition 4.2.1. Let R, S be rings. An R-S-**bimodule** is an abelian group M being both a left R-module and a right S-module, satisfying:

$$r(ms) = (rm)s$$

for all $m \in M, r \in R, s \in S$.

Example 4.2.2. Any left R-module is an R-$\mathbb{Z}$-bimodule, and any right R-module is a $\mathbb{Z}$-R-bimodule.

Example 4.2.3. When R is commutative, any R-module can be seen as an R-R-bimodule.

Proposition 4.2.4. Let Q, R, S be rings, M be a Q-R-bimodule, and N be a R-S-bimodule. Then $M \otimes_R N$ is a Q-S-bimodule.

Proof. Let $q \in Q$ and $s \in S$. Then $f : m \mapsto qm$ is a homomorphism $M \to M$ and $g : n \mapsto ns$ is a homomorphism $N \to N$. Then $f \otimes \mathrm{id}_N$ gives a left multiplication on $M \otimes_R N$ and $\mathrm{id}_M \otimes g$ gives a right multiplication on $M \otimes_R N$, which satisfies $(f \otimes \mathrm{id}_N) \circ (\mathrm{id}_M \otimes g) = f \otimes g = (\mathrm{id}_M \otimes g) \circ (f \otimes \mathrm{id}_N)$. ■

Definition 4.2.5. If R is a commutative ring and M, N, A are R-modules, a map $f : M \times N \to A$ is called R-**bilinear** if it is R-biadditive and also

$$f(rx, y) = f(x, ry) = rf(x, y)$$

for all $x \in M$, $y \in N$ and $r \in R$.

Proposition 4.2.6. Let R be a commutative ring and A, B be R-modules. Then $A \otimes_R B$ is an R-module and $h : A \times B \to A \otimes_R B$ is R-bilinear.

Further, for any R-bilinear map $g : A \times B \to C$, there exists an R-homomorphism $\hat{g} : A \otimes_R B \to C$ such that $g = \hat{g} \circ h$.

Proof. We view A, B as R-R-bimodules, then we easily see that $A \otimes_R B$ is also an R-R-bimodule (i.e., an R-module) with (left) multiplication given by $(a \mapsto ra) \otimes \mathrm{id}_B$, hence $r(a \otimes b) = (ra) \otimes b = a \otimes (rb)$ and h is R-bilinear.

Suppose $g : A \times B \to C$ is an R-bilinear map. Then g is R-biadditive and g induces a $\mathbb{Z}$-homomorphism $\hat{g} : A \otimes_R B \to C$ such that $g = \hat{g} \circ h$. We only need to show that $\hat{g}$ is also an R-homomorphism. Let $r \in R$. Then $\hat{g}(r(a \otimes b)) = \hat{g}((ra) \otimes b) = g(ra, b) = rg(a, b) = r\hat{g}(a \otimes b)$. ∎

4.3. Further Properties

Proposition 4.3.1. If R is a ring, M is a right R-module and N is a left R-module, then there is a natural $\mathbb{Z}$-isomorphism

$$\begin{aligned} \tau : M \otimes_R N &\to N \otimes_{R^{\mathrm{op}}} M \\ m \otimes n &\mapsto n \otimes m \end{aligned}$$

Proof. This follows from the fact that a map $f : M \times N \to A$ is R-biadditive if and only if the map $g : N \times M \to A$ defined by $g(n, m) = f(m, n)$ is R^{op}-biadditive. ∎

Corollary 4.3.2. If R is a commutative ring and M, N are R-modules, then there is a natural R-isomorphism

$$\begin{aligned} \tau : M \otimes_R N &\to N \otimes_R M \\ m \otimes n &\mapsto n \otimes m \end{aligned}$$

Proposition 4.3.3. Given right R-module A, R-S-bimodule B, and left S-module C, there is an isomorphism

$$\theta : A \otimes_R (B \otimes_S C) \cong (A \otimes_R B) \otimes_S C$$

given by $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$.

Proof. (Rotman 2009, Proposition 2.57). They are both solutions to the universal mapping problem of **triadditive functions**, but the solution is unique. ∎

Corollary 4.3.4. Let R be a commutative ring, and let $M_1, M_2, \ldots, M_n$ be R-modules. Let $\sigma \in S_n$ (where S_n is the symmetric group of degree n), then

$$(\ldots(M_1 \otimes_R M_2) \otimes_R \ldots \otimes_R M_n) \cong \left(\ldots\left(M_{\sigma(1)} \otimes_R M_{\sigma(2)}\right) \otimes_R \ldots \otimes_R M_{\sigma(n)}\right)$$

Proof. Notice that both solve the universal mapping problem of R-n-linear functions. ■

Proposition 4.3.5. Given ring R and left R-module M, there is a natural R-isomorphism

$$\begin{aligned} \varphi_M : R \otimes_R M &\to M \\ r \otimes m &\mapsto rm \end{aligned}$$

Proof. $f : R \times M \to M$ defined by $(r, m) \mapsto rm$ is R-biadditive and thus induces an R-homomorphism $\varphi : R \otimes_R M \to M$ with $r \otimes m \mapsto rm$. Now $g : M \to R \otimes_R M$ defined by $g : m \mapsto 1 \otimes m$ satisfies that φg and $g\varphi$ are identity maps, so φ is an R-isomorphism. ■

4.4. Monoidal Categories and k-algebras

Definition 4.4.1. A **monoidal category** is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ associative up to a natural isomorphism, and an object I that is both a left and right identity for $\otimes$ up to a natural isomorphism.

Proposition 4.4.2. Let R be a commutative ring, then the category $(R\text{-}\mathbf{Mod}, \otimes_R)$ is a monoidal category. In particular, for a field k, the category $(\mathbf{Vect}_k, \otimes_k)$ is a monoidal category.

Proof. The identity for $\otimes_R$ in R-$\mathbf{Mod}$ is clearly given by R. ■

Definition 4.4.3. A **monoid object** in a monoid category $(\mathcal{C}, \otimes, I)$ is an object M with two morphisms:

- $\mu : M \otimes M \to M$ called **multiplication**,
- $\eta : I \to M$ called **unit**,

such that the following diagrams commute:

$$\begin{array}{ccccc} (M \otimes M) \otimes M & \xrightarrow{\alpha} & M \otimes (M \otimes M) & \xrightarrow{\mathrm{id}_M \otimes \mu} & M \otimes M \\ \downarrow{\scriptstyle \mu \otimes \mathrm{id}_M} & & & & \downarrow{\scriptstyle \mu} \\ M \otimes M & & \xrightarrow{\qquad\qquad\mu\qquad\qquad} & & M \end{array}$$

$$\begin{array}{ccccc} I \otimes M & \xrightarrow{\eta \otimes \mathrm{id}_M} & M \otimes M & \xleftarrow{\mathrm{id}_M \otimes \eta} & M \otimes I \\ & {\scriptstyle \lambda} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle \rho} & \\ & & M & & \end{array}$$

where α, λ, ρ are natural isomorphisms for the associativity, the left identity and the right identity, respectively.

Definition 4.4.4. Let k be a field. A (unital associative) k**-algebra** is a monoid object in $(\mathbf{Vect}_k, \otimes_k)$.

Remark 4.4.5. Let M be a k-algebra, then M is a k-vector space equipped with bilinear multiplication $\mu : M \otimes_k M \to M$ and unit $\eta : k \to M$ which sends $1 \in k$ to $i(1) \in M$, the multiplicative unit.

An equivalent definition: M is both a k-vector space and a unital ring, where the ring multiplication satisfies

$$a(xy) = (ax)y = x(ay)$$

for all $a \in k$ and $x, y \in M$.

Remark 4.4.6. $(\mathbf{Vect}_k, \otimes_k)$ is a **symmetric monoidal category**, where the tensor product is commutative.

Remark 4.4.7. In $\mathbf{Set}$,

$$\mathrm{Hom}(X \times Y, Z) = \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)).$$

$\mathbf{Set}$ is a **cartesian monoidal category**, where the categorical product is the same as the tensor product.

Remark 4.4.8. For vector spaces V, W over field k,

$$\dim(V \oplus W) = \dim V + \dim W, \quad \dim(V \otimes W) = \dim V \cdot \dim W.$$

An alternative definition of the tensor product: let V be a vector space with basis $\{v_i\}_{i \in I}$ and W with $\{w_j\}_{j \in J}$ and define $V \otimes_k W$ to have basis $\{v_i \otimes w_j\}_{i \in I, j \in J}$. This definition relies on the choice of basis and can be inconvenient when we have to change basis.

4.5. Tensor-hom Adjunction

Proposition 4.5.1. If B is an R-S-bimodule and C is a right S-module, then $\mathrm{Hom}_S(B, C)$ is a right R-module.

Proof. Take any $f \in \mathrm{Hom}_S(B, C)$ and $r \in R$, define right multiplication $fr : B \to C$ by $fr(b) = f(rb)$. Then we see that $(fr)(bs) = f(rbs) = f(rb)s = (fr)(b)s$ which indicates that fr still an S-homomorphism $B \to C$. ∎

Then this makes $\mathrm{Hom}_S(B, -)$ a functor from $\mathbf{Mod}$-S to $\mathbf{Mod}$-R.

Theorem 4.5.2. Let R, S be rings. Let A be a right R-module, B be a R-S-bimodule, and C be a right S-module. Then we have a canonical isomorphism

$$\tau : \mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow{\sim} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

where for $f : A \otimes_R B \to C$, $a \in A$, and $b \in B$,

$$\tau(f)(a)(b) = f(a \otimes b)$$

Proof. τ is a group homomorphism because for any $a \in A$ and $b \in B$,

$$\tau(f+g)(a)(b) = (f+g)(a \otimes b) = f(a \otimes b) + g(a \otimes b) = \tau(f)(a)(b) + \tau(g)(a)(b)$$

and hence $\tau(f+g) = \tau(f) + \tau(g)$.

τ is injective because if $\tau(f) = 0$, then $f(a \otimes b) = \tau(f)(a)(b) = 0$ for all $a \in A$ and $b \in B$. Thus $f = 0$ since it vanishes on all generators of $A \otimes_R B$.

It remains to be shown that τ is surjective. Take any R-homomorphism $g : A \to \mathrm{Hom}_S(B, C)$, define $\varphi : A \times B \to C$ by $\varphi(a, b) = g(a)(b)$. Now it is easy to check that φ is R-biadditive and hence there exists a group homomorphism $\hat{\varphi} : A \otimes_R B \to C$ such that $\hat{\varphi}(a \otimes b) = \varphi(a, b) = g(a)(b)$ for all $a \in A$ and $b \in B$. Therefore $g = \tau(\hat{\varphi})$ and τ is surjective.

Verifying the naturality of τ is omitted. ∎

Corollary 4.5.3. Let R, S be rings and let B be a R-S-bimodule. We have an adjunction

$$(- \otimes_R B) \dashv \mathrm{Hom}_S(B, -)$$

where $(- \otimes_R B) : \mathbf{Mod}\text{-}R \to \mathbf{Mod}\text{-}S$ and $\mathrm{Hom}_S(B, -) : \mathbf{Mod}\text{-}S \to \mathbf{Mod}\text{-}R$.

Corollary 4.5.4. The functor $(- \otimes_R B) : \mathbf{Mod}\text{-}R \to \mathbf{Mod}\text{-}S$ preserves colimits. In particular, it preserves cokernels and is thus right exact; it also preserves direct sums.

Theorem 4.5.5. Let A be a left R-module, B be a S-R-bimodule, and C be a left S-module, then there is a canonical isomorphism

$$\mathrm{Hom}_S(B \otimes_R A, C) \xrightarrow{\sim} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

Thus $(B \otimes_R -) : R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$ and $\mathrm{Hom}_S(B, -) : S\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ form an adjunction

$$(B \otimes_R -) \dashv \mathrm{Hom}_S(B, -)$$

Hence $(B \otimes_R -)$ preserves colimits and in particular is right exact.

4.6. Computations

Example 4.6.1. Let R be a commutative ring. Given R-modules N and M, suppose we want to calculate $N \otimes_R M$, then we can pick the relations and generators of N:

$$R^{\oplus J} \to R^{\oplus I} \to N \to 0$$

Consider $R^{\oplus J} \to R^{\oplus I}$, this homomorphism between free modules can be represented by a (possibly infinite) matrix $a_{ij} \in R$. Therefore, we can write

$$N = \operatorname{Coker}\left(R^{\oplus J} \overset{(a_{ij})}{\to} R^{\oplus I} \right)$$

Notice that the same matrix can also act as $M^{\oplus J} \to M^{\oplus I}$, hence

$$N \otimes_R M = \operatorname{Coker}\left(R^{\oplus J} \overset{(a_{ij})}{\to} R^{\oplus I} \right) \otimes_R M \cong \operatorname{Coker}\left(R^{\oplus J} \otimes_R M \overset{(a_{ij})}{\to} R^{\oplus I} \otimes_R M \right)$$

$$\cong \operatorname{Coker}\left((R \otimes_R M)^{\oplus J} \overset{(a_{ij})}{\to} (R \otimes_R M)^{\oplus I} \right) \cong \operatorname{Coker}\left(M^{\oplus J} \overset{(a_{ij})}{\to} M^{\oplus I} \right)$$

Example 4.6.2. Suppose I is an ideal of R generated by $\{x_i\}_{i \in J}$, then we have short exact sequence

$$0 \to I \to R \to R/I \to 0$$

On the other hand,

$$R^{\oplus J} \to I \to 0$$

and thus

$$R^{\oplus J} \overset{(x_j)}{\to} R \to R/I \to 0$$

Let M be a left R-module, then

$$(R/I) \otimes_R M \cong \operatorname{Coker}\left(M^{\oplus J} \overset{(x_j)}{\to} M \right) = M/IM$$

Example 4.6.3. The **localisation** of a commutative ring R at element x is defined as $R[t]/(tx - 1)$, denoted as $R[x^{-1}]$.

For R-module M, we have

$$R[x^{-1}] \otimes_R M \cong M[x^{-1}]$$

Proof. Notice that $R[t] \otimes_R M = M[t]$, because $R[t] \cong \bigoplus_{i=1}^{\infty} R$ in R-**Mod**. ∎

5. Enough Projectives and Injectives

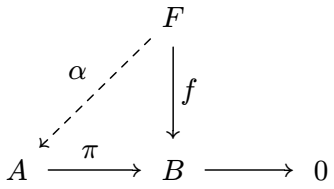
Definition 5.1. An abelian category $\mathcal{A}$ is said to **have enough projectives** (resp. **injectives**) if for any object M there is an epimorphism $P \to M \to 0$ where P is projective (resp. a monomorphism $0 \to M \to I$ where I is injective).

For most of our homological algebra to work, an abelian category needs to have enough projectives and injectives. We will show that R-**Mod** has enough projectives and injectives.

5.1. R-Mod has Enough Projectives

Lemma 5.1.1. Free R-modules are projective.

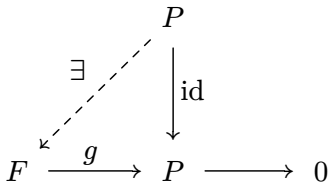
Proof. Let $F = \bigoplus_{i \in I} Rx_i$ be a free R-module with basis $\{x_i : i \in I\}$. Suppose $\pi : A \to B$ is an epimorphism and $f : F \to B$ is a morphism, as in the following diagram:



Since π is surjective, for each i there is some $a_i \in A$ with $\pi(a_i) = f(x_i)$. Define map $\alpha : F \to A$ by $\alpha(x_i) = a_i$ and we have $f = \pi \circ \alpha$. ∎

Proposition 5.1.2. P is a projective R-module if and only if P is a direct summand of a free module.

Proof. Assume P is a projective. Then we can always find a free module $F = R^{\oplus I}$ such that $g : F \to P$ is onto. Using the lifting property,

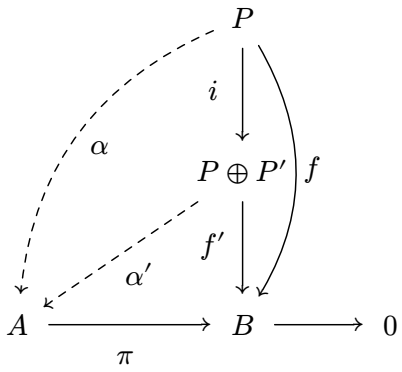


So there exists a section $P \to F$ in the short exact sequence

$$0 \to K \to F \to P \to 0$$

and hence $F \cong K \oplus P$, where $K = \mathrm{Ker}(g)$. This shows that P is a direct summand of a free module.

Now we show a direct summand of a free module is projective. Suppose that P is a direct summand of a free module. Then there exists some R-module P' such that $P \oplus P'$ is free. Let $\pi : A \to B$ be a surjection and let $f : P \to B$ be some map. Let $f' : P \oplus P' \to B$ be the map $f'(p, p') = f(p)$. Since $P \oplus P'$ is free, hence projective, f' has a lift $\alpha' : P \oplus P' \to A$. Now define $\alpha : P \to A$ by $\alpha(p) = \alpha'(p, 0)$ and it lifts f, showing that P is projective.



■

Corollary 5.1.3. R-**Mod** has enough projectives.

Proof. For any module M, we can find a free (and thus projective) module F with a surjection $F \to M \to 0$. ■

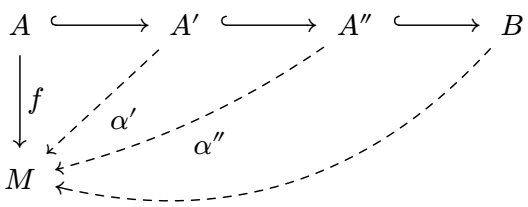
5.2. Ab has Enough Injectives

Lemma 5.2.1 (Baer's Criterion). A right (resp. left) R-module M is injective if and only if for every right (resp. left) ideal I of R, every module homomorphism $I \to M$ can be extended to $R \to M$.

Proof. (Monnet and Kremnitzer 2021, Theorem 5.8) and (Rotman 2009, Theorem 3.30).

"⇒". Since any right ideal I is a submodule of R, we can extend $I \to M$ to $R \to M$ simply by the definition of injectivity of M.

"⇐".



Fix some injection $i : A \to B$ of R-modules, and some map $f : A \to M$. Without loss of generality, assume that $A \subseteq B$ and i is the inclusion. We would like to extend f to some map $B \to M$.

[Construction of A' and $\alpha' : A' \to M$.] Let Σ be the set whose elements are R-module maps $\alpha' : A' \to M$, where $A \subseteq A' \subseteq B$ and α' extends f. We may give this set a partial order by saying that $\alpha' \leq \alpha''$ when $A' \subseteq A''$ and α'' extends α'. Suppose that $\alpha_1 \leq \alpha_2 \leq \ldots$ is an ascending chain in Σ, with corresponding modules $A_1 \subseteq A_2 \subseteq \ldots$. Let $A' = \bigcup A_n$, and define $\alpha' : A' \to M$ by $\alpha'(a) = \alpha_i(a)$ for $a \in A_i$. It is easy to check that α' is a well-defined element of Σ, and it is an upper bound on the chain (in other words, we take the colimit of the chain). Since Σ is a partially ordered set in which every ascending chain has an upper bound, by Zorn's Lemma Σ has a maximal element, which we call $\alpha' : A' \to M$.

To show that M is injective, we need to show that $A' = B$, since we then have an extension α of f to B.

[Construction of $\varphi : R \to M$.] Suppose that $A' \neq B$. Let $b \in B \setminus A'$. Let

$$I = \{r \in R : br \in A'\}$$

Then I is a right ideal of R, and we have a map

$$\begin{aligned} \tilde{\varphi} : I &\to M \\ r &\mapsto \alpha'(br). \end{aligned}$$

By assumption, this extends to a map $\varphi : R \to M$.

[Construction of $\alpha'' : A'' \to M$.] Define

$$A'' = A' + Rb = \{a + rb : a \in A', r \in R\} \subseteq B$$

We claim that there is a well-defined map

$$\begin{aligned} \alpha'' : A'' &\to M \\ a + br &\mapsto \alpha'(a) + \varphi(r), \end{aligned}$$

where $a \in A'$ and $r \in R$. To see that this is well-defined, suppose that $a + br = a' + br'$ where $a, a' \in A'$ and $r, r' \in R$. Then

$$a - a' = b(r' - r) \in A' \cap bR.$$

From this we see $r - r' \in I$, and then we have

$$\varphi(r' - r) = \alpha'(b(r' - r)) = \alpha'(a - a').$$

Therefore, it follows that $\alpha'(a) + \varphi(r) = \alpha'(a') + \varphi(r')$ so α'' is well-defined. But then α'' strictly extends α', contradicting maximality of α'. Hence $A' = B$, so M is injective. ■

Definition 5.2.2. Let R be an integral domain. A R-module M is called **divisible** if, for all $r \in R \setminus \{0\}$, every element m of M can be "divided" by r, in the sense that there exists an element m' in M such that $m = rm'$. Equivalently, the multiplication by any non-zero $r \in R$ defines a surjective map from M to M.

Corollary 5.2.3. If R is a PID, then an R-module M is injective if and only if it is divisible.

Proof. (Rotman 2009, Corollary 3.35) and (Monnet and Kremnitzer 2021, Corollary 5.9). Let M be an injective R-module, and let $m \in M$ and $r \in R \setminus \{0\}$. Set $J = rR$ (which is an ideal of R) and define $f : J \to M$ by $f(r) = m$. By Baer's Criterion, we may extend f to a homomorphism $\tilde{f} : R \to M$. Then

$$m = f(r) = \tilde{f}(r) = \tilde{f}(r \cdot 1) = r \cdot \tilde{f}(1).$$

So taking $m' = \tilde{f}(1)$, we see that M is divisible.

Suppose conversely that M is a divisible R-module. Let J be an ideal of R and let $f : J \to M$ be a module homomorphism. If J is the zero ideal, then trivially we may extend f to the zero homomorphism $R \to M$. Assume that J is nonzero. Since R is a PID, we have $J = Rr$ for some nonzero $r \in J$. Let $m = f(r)$. Then since M is divisible, there is some $m' \in M$ such that $m = rm'$. Define $\tilde{f} : R \to M$ by $\tilde{f}(1) = m'$. Clearly $\tilde{f}$ is an extension of f, so M is injective by Baer's Criterion. ■

Corollary 5.2.4. In $\mathbf{Ab}$, $\mathbb{Q}, \mathbb{Z}_{p^\infty} = \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}$ are injective.

Remark 5.2.5. (Weibel 1994, Example 2.3.3). Every injective abelian group $I = I_{\text{tor}} \oplus I_{\text{free}}$, where I_{free} (the torsion-free part) is a $\mathbb{Q}$-vector space and I_{tor} (the torsion part) is a direct sum of copies of $\mathbb{Z}_{p^\infty}$. In particular, $\mathbb{Q}/\mathbb{Z} = \oplus_p \mathbb{Z}_{p^\infty}$.

Lemma 5.2.6. Direct sums of projectives are projectives. Dually, products of injectives are injectives.

Proof. Suppose $\{P_i : i \in I\}$ is a family of projective modules. Then for each $i \in I$, by Proposition 5.1.2 we can write $F_i = P_i \oplus Q_i$ for some free R-module F_i and R-module Q_i. Then

$$\bigoplus_{i \in I} F_i = \bigoplus_{i \in I} P_i \oplus \bigoplus_{i \in I} Q_i$$

Since $\bigoplus_{i \in I} F_i$ is also a free module, $\bigoplus_{i \in I} P_i$ is also projective. ■

Lemma 5.2.7. Let A be an abelian group. Then for any non-zero $a \in A$, there exists a group homomorphism $\varphi : A \to \mathbb{Q}/\mathbb{Z}$ such that $\varphi(a) \neq 0$.

Proof. By the injectivity of $\mathbb{Q}/\mathbb{Z}$, it suffices to find a group homomorphism $\psi : \langle a \rangle \to \mathbb{Q}/\mathbb{Z}$ and then extend ψ to $\varphi : A \to \mathbb{Q}/\mathbb{Z}$. To obtain such ψ, it suffices to give an element $\psi(a) \in \mathbb{Q}/\mathbb{Z}$. We consider the order of a in A:

- If $|a| = \infty$, then we can set $\psi(a)$ as any nonzero element of $\mathbb{Q}/\mathbb{Z}$;
- If $|a| = m > 1$, then we set $\psi(a) = \frac{1}{m} + \mathbb{Z}$.

■

Proposition 5.2.8. $\mathbf{Ab}$ has enough injectives.

Proof. Define a map

$$\begin{aligned} I : \mathbf{Ab} &\to \mathbf{Ab} \\ A &\mapsto \prod_{\mathrm{Hom}_{\mathbf{Ab}}(A,\mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

For any $A \in \mathbf{Ab}$, $I(A)$ is injective as a product of injectives $\mathbb{Q}/\mathbb{Z}$. Consider canonical map

$$\begin{aligned} e_A : A &\to I(A) \\ a &\mapsto (\varphi(a))_{\varphi \in \mathrm{Hom}_{\mathbf{Ab}}(A,\mathbb{Q}/\mathbb{Z})}, \end{aligned}$$

where, since $I(A)$ is a product, we need to define for each $\varphi \in \mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})$ the component $e_{a,\varphi} : A \to \mathbb{Q}/\mathbb{Z}$, which we just define to be φ itself. Note that e_A is an injective map by Lemma 5.2.7. Thus we have $0 \to A \xrightarrow{e_A} I(A)$ with $I(A)$ injective for any $A \in \mathbf{Ab}$, showing that $\mathbf{Ab}$ has enough injectives. ■

5.3. R-Mod has Enough Injectives

Proposition 5.3.1. If an additive functor $R : \mathcal{B} \to \mathcal{A}$ between abelian categories is right adjoint to an exact functor and I is injective in $\mathcal{B}$, then $R(I)$ is injective in $\mathcal{A}$. Dually, if an additive functor $L : \mathcal{A} \to \mathcal{B}$ is left adjoint to an exact functor and P is projective in $\mathcal{A}$, then $L(P)$ is projective in $\mathcal{B}$.

Proof. (Monnet and Kremnitzer 2021, Lemma 5.25) and (Weibel 1994, Proposition 2.3.10). We want to show that

$$\mathrm{Hom}_{\mathcal{A}}(-, R(I))$$

is exact. We have

$$\mathrm{Hom}_{\mathcal{A}}(-, R(I)) \cong \mathrm{Hom}_{\mathcal{B}}(L(-), I)$$

but L is exact by assumption and $\mathrm{Hom}_{\mathcal{B}}(-, I)$ is exact because I is injective in $\mathcal{B}$, so $\mathrm{Hom}_{\mathcal{B}}(L(-), I)$ is a composition of exact functors and thus exact. ■

With this proposition, we can prove that an abelian category has enough projectives or injectives by constructing adjunctions.

Corollary 5.3.2. If I is an injective abelian group, then $\mathrm{Hom}_{\mathbf{Ab}}(R, I)$ is an injective right R-module.

Proof. By Proposition 4.5.1, $\mathrm{Hom}_{\mathbf{Ab}}(R, I)$ is indeed a right R-module. Note that $\mathrm{Hom}_{\mathbf{Ab}}(R, -)$ is right adjoint to $(- \otimes_R R)$, which is simply the forgetful functor $\mathbf{Mod}\text{-}R \to \mathbf{Ab}$ and is thus exact. Therefore $\mathrm{Hom}_{\mathbf{Ab}}(R, I)$ is injective in R-$\mathbf{Mod}$. ■

Example 5.3.3. $\mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$ is injective.

Proposition 5.3.4. R-**Mod** has enough injectives.

Proof. Define map

$$\begin{aligned} I : R\text{-}\mathbf{Mod} &\to R\text{-}\mathbf{Mod} \\ M &\mapsto \prod_{\mathrm{Hom}_R(M,\, \mathrm{Hom}_{\mathbf{Ab}}(R,\mathbb{Q}/\mathbb{Z}))} \mathrm{Hom}_{\mathbf{Ab}}(R,\mathbb{Q}/\mathbb{Z}) \end{aligned}$$

For any left R-module M, $I(M)$ is injective as a product of injectives, and there is a canonical morphism

$$\begin{aligned} e_M : M &\to I(M) \\ m &\mapsto (\varphi(m))_{\varphi \in \mathrm{Hom}_R(M,\, \mathrm{Hom}_{\mathbf{Ab}}(R,\mathbb{Q}/\mathbb{Z}))} \end{aligned}$$

We would like to show that e_M is an injective function. We only need to show that for any $0 \neq m \in M$, there exists $\varphi : M \to \mathrm{Hom}_{\mathbf{Ab}}(R,\mathbb{Q}/\mathbb{Z})$ such that $\varphi(m) \neq 0$. Notice that we have

$$\varphi \in \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbf{Ab}}(R,\mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbf{Ab}}(M,\mathbb{Q}/\mathbb{Z})$$

as before. Hence we only need to find some $\varphi : M \to \mathbb{Q}/\mathbb{Z}$ in $\mathbf{Ab}$ so that $\varphi(m) \neq 0$, which is given by Lemma 5.2.7. ∎

6. Chain Complexes

6.1. Definitions

Let $\mathcal{A}$ be an abelian category.

Definition 6.1.1. A **chain complex** $C_\bullet$ in $\mathcal{A}$ is a family $\{C_n\}_{n\in\mathbb{Z}}$ of objects in $\mathcal{A}$ with morphisms $d_n : C_n \to C_{n-1}$ such that $d_n \circ d_{n-1} = 0$, where d_n are called **differentials**. The n-**cycles** of $C_\bullet$ are defined as

$$Z_n(C) := \operatorname{Ker} d_n$$

and the n-**boundaries** are defined as

$$B_n(C) := \operatorname{Im} d_{n+1}.$$

Since $d_n \circ d_{n-1} = 0$, we have

$$B_n(C) \hookrightarrow Z_n(C) \hookrightarrow C_n$$

(as subobjects) for all n.

The n-**th homology** is defined as

$$H_n(C) := \operatorname{Coker}(B_n(C) \hookrightarrow Z_n(C)).$$

Notation 6.1.2. We often omit the subscript in d_n and simply write d, so $d_n \circ d_{n-1} = 0$ becomes $d^2 = 0$. To emphasise that d belongs to the chain complex $C_\bullet$, we would write either d_C, or $d_n^{(C)}$ if we also need to explicitly specify the index. We sometimes also omit the dot in $C_\bullet$ and simply write C. We might write $Z_n = Z_n(C)$ and $B_n = B_n(C)$.

Remark 6.1.3. In the case of R-**Mod**, an n-**cycle** in C_n is an element $x \in C_n$ such that $d(x) = 0$, and an n-**boundary** in C_n is an element $y \in C_n$ such that there exists $c' \in C_{n+1}$ such that $d(c') = y$. An n-boundary must be an n-cycle because $d^2 = 0$. The n-th homology becomes a quotient module[3],

$$H_n(C) = \frac{Z_n}{B_n} = \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}$$

An element in $H_n(C)$ can be written as $x + B_n$, or simply $[x]$, for some n-cycle x.

Remark 6.1.4. It is helpful to keep in mind two defining short exact sequences:

$$0 \to Z_n \to C_n \xrightarrow{d_n} B_{n-1} \to 0,$$
$$0 \to B_n \hookrightarrow Z_n \to H_n \to 0.$$

From these we know that exact functors preserve homology as they preserve short exact sequences.

[3] The slogan is that "homology is cycles modulo boundaries" or even "homology is kernel modulo image".

Definition 6.1.5. We can form a category $\mathrm{Ch}(\mathcal{A})$ where objects are chain complexes and morphisms are **chain maps** $u_\bullet : C_\bullet \to D_\bullet$ which commutes with differentials

$$ud = du.$$

Namely, for all $n \in \mathbb{Z}$,

$$\begin{array}{ccc} C_n & \xrightarrow{d} & C_{n-1} \\ \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle u_{n-1}} \\ D_n & \xrightarrow{d} & D_{n-1} \end{array}$$

commutes.

Proposition 6.1.6. $\mathrm{Ch}(\mathcal{A})$ is an abelian category if $\mathcal{A}$ is an abelian category.

Proof. (Rotman 2009, Proposition 5.100). ■

Proposition 6.1.7. A chain map $u_\bullet : C_\bullet \to D_\bullet$ induces a morphism $H_n(u) : H_n(C) \to H_n(D)$.

Proof. It suffices to assume $\mathcal{A} = R\text{-}\mathbf{Mod}$. First we show that $u_n : C_n \to D_n$ sends boundaries to boundaries. Take boundary $b \in C_n$, then there exists $c \in C_{n+1}$ such that $d(c) = b$. Thus $u(b) = ud(c) = du(c)$, showing that $u(b)$ is a boundary in D_n. Next we show that $u_n : C_n \to D_n$ sends cycles to cycles. Take cycle $z \in C_n$ such that $d(z) = 0$. Then $du(z) = ud(z) = u(0) = 0$, showing that $u(z)$ is a cycle in D_n. Therefore, u_n induces a function $H_n(C) \to H_n(D)$. ■

Corollary 6.1.8. $H_n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ is an additive functor.

Definition 6.1.9. A chain map $C_\bullet \to D_\bullet$ is called a **quasi-isomorphism** if the induced maps $H_n(C) \to H_n(D)$ are isomorphisms for all n.

Proposition 6.1.10. The followings are equivalent:
- $C_\bullet$ is exact at every C_n;
- $C_\bullet$ is **acyclic**, i.e., $H_n(C) = 0$ for all n;
- $0 \to C_\bullet$ is a quasi-isomorphism.

Definition 6.1.11. A **cochain complex** $C_\bullet$ in $\mathcal{A}$ is a family $\{C^n\}_{n\in\mathbb{Z}}$ of objects in $\mathcal{A}$ with morphisms $d^n : C^n \to C^{n+1}$ such that $d^n \circ d^{n+1} = 0$, where d^n are called **differentials**. The n **-cocycles** of $C^\bullet$ are

$$Z^n(C) := \operatorname{Ker} d^n$$

and the n**-coboundaries** are

$$B^n(C) := \operatorname{Im} d^{n-1}.$$

We have

$$B^n(C) \hookrightarrow Z^n(C) \hookrightarrow C^n$$

(as subobjects) for all n.

The n**-th cohomology** are defined as

$$H^n(C) := \operatorname{Coker}(B^n(C) \hookrightarrow Z^n(C)).$$

We also define **cochain maps** similarly as before.

Example 6.1.12. (Weibel 1994, Application 1.1.4). Let X be a topological space, and let $C_k = C_k(X)$ be the free R-module on the set of continuous maps from the standard k-simplex Δ_k to X. Restriction to the i-th face of $\Delta_k (0 \le i \le k)$ transforms a map $\Delta_k \to X$ into a map $\Delta_{k-1} \to X$, and induces an R-module homomorphism ∂_i from C_k to C_{k-1}. The alternating sums $d = \sum (-1)^i \partial_i$ (from C_k to C_{k-1}) assemble to form a chain complex

$$\cdots \xrightarrow{d} C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \to 0$$

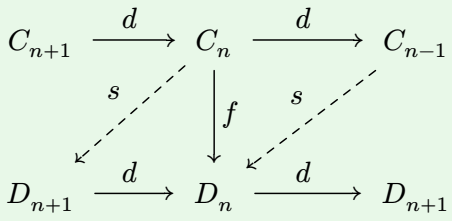
called the **singular chain complex** of X. The n-th homology module of $C_\bullet(X)$ is called the n -th singular homology of X (with coefficients in R) and is written $H_n(X; R)$.

6.2. Chain Homotopy

Definition 6.2.1. A chain map $f : C_\bullet \to D_\bullet$ is **null homotopic** if there are maps $s_n : C_n \to D_{n+1}$ such that $f = ds + sd$, or more rigorously,

$$f_n = d_{n+1}s_n + s_{n+1}d_n$$

for all n.



(Only the solid lines commute.) We denote $f \sim 0$ in this case.

Definition 6.2.2. Two chain maps f and g from $C_\bullet$ to $D_\bullet$ are **chain homotopic** if $f - g$ is null homotopic. We denote $f \sim g$.

Remark 6.2.3. $f \sim g \Leftrightarrow f - g \sim 0 \Leftrightarrow f - g = sd + ds$.

Lemma 6.2.4. Suppose that chain maps $f, g : C_\bullet \to D_\bullet$ are chain homotopic. Then the induced maps $f_*, g_* : H_n(C) \to H_n(D)$ are equal. In particular, if $f : C_\bullet \to D_\bullet$ is null homotopic, then $f_* = 0 : H_n(C) \to H_n(D)$.

Proof. (Monnet and Kremnitzer 2021, Lemma 2.32), (Weibel 1994, Lemma 1.4.5). Let h be a chain homotopy from f to g. We have

$$f_n - g_n = s_{n-1} \circ d_n^{(C)} + d_{n+1}^{(D)} \circ s_n$$

for each n. Let $x \in H_n(C)$. Then $x = [c]$ for some cycle $c \in Z_n C$. We have

$$\begin{aligned} f_*(x) - g_*(x) &= [f_n(c) - g_n(c)] \\ &= \left[s_{n-1} \circ d_n^{(C)}(c) + d_{n+1}^{(D)} \circ s_n(c)\right] \\ &= \left[d_{n+1}^{(D)} \circ s_n(c)\right] \\ &= 0, \end{aligned}$$

The third equality is because c is an n-cycle in C and last equality is because $d_{n+1}^{(D)} \circ s_n(c)$ is an n-boundary in D. ∎

Corollary 6.2.5. If the chain map $\mathrm{id} : C_\bullet \to C_\bullet$ is null homotopic, then $C_\bullet$ is acyclic.

Definition 6.2.6. (Weibel 1994, Translation 1.2.8). If $C = C_\bullet$ is a chain complex (resp. cochain complex) and p an integer, we form a new complex $C[p]$ as follows:

$$C[p]_n = C_{n+p} \quad (\text{resp. } C[p]^n = C^{n-p})$$

with differential $(-1)^p d$. We call $C[p]$ the p-**th translate** of C. The way to remember the shift is that the degree 0 part of $C[p]$ is C_p. The sign convention is designed to simplify notation later on. Note that translation shifts homology:

$$H_n(C[p]) = H_{n+p}(C) \quad (\text{resp. } H^n(C[p]) = H^{n-p}(C)).$$

We make translation into a functor $[p] : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ by shifting indices on chain maps. That is, if $f : C \to D$ is a chain map, then $f[p]$ is the chain map given by the formula

$$f[p]_n = f_{n+p} \quad (\text{resp. } f[p]^n = f^{n-p}).$$

6.3. Exact Sequences

Recall that if $\mathcal{A}$ is an abelian category, then $\mathrm{Ch}(\mathcal{A})$ is also an abelian category. Therefore, we can form short exact sequences with chain complexes, and it turns out that they naturally induce long exact sequences in (co)homology.

Definition 6.3.1. For chain complexes $A_\bullet, B_\bullet, C_\bullet$,

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a **short exact sequence** if $0 \to A_n \to B_n \to C_n \to 0$ is a short exact sequence for all n.

Theorem 6.3.2. If $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$ is a short exact sequence of chain complexes, then there is a natural map for each n

$$\partial_n : H_n(C) \to H_{n-1}(A),$$

which we call the **connecting homomorphism**, making

$$\ldots \to H_n(B) \to H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \to H_{n-1}(B) \to \ldots$$

a long exact sequence. Further, ∂_n is explicitly given by the well-defined expression

$$\partial_n = f^{-1} d_B g^{-1}.$$

If $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$ is a short exact sequence of cochain complexes, then we have the connecting homomorphism

$$\partial^n : H^n(C) \to H^{n+1}(A),$$

where the induced long exact sequence is

$$\ldots \to H_n(B) \to H_n(C) \xrightarrow{\partial^n} H_{n+1}(A) \to H_{n+1}(B) \to \ldots$$

and

$$\partial^n = f^{-1} d^B g^{-1}.$$

Proof. Again, we assume the context of R-**Mod**. This is an application of the Snake Lemma 3.7.12.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_A} & & \downarrow{\scriptstyle d_B} & & \downarrow{\scriptstyle d_C} & & \\ 0 & \longrightarrow & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & 0 \end{array}$$

Using the Snake Lemma, if we write the cokernels (and shift up the index by 1), we get

$$\frac{A_n}{dA_{n+1}} \to \frac{B_n}{dB_{n+1}} \to \frac{C_n}{dC_{n+1}} \to 0$$

is exact, where $dA_{n+1} = \operatorname{Im} d$; if we write the kernels, we get

$$0 \to Z_{n-1}(A) \to Z_{n-1}(B) \to Z_{n-1}(C)$$

is also exact. Notice that $dA_n \subseteq Z_{n-1}(A)$, so we can use d to connect the rows again:

$$\begin{array}{ccccccccc} & & \frac{A_n}{dA_{n+1}} & \longrightarrow & \frac{B_n}{dB_{n+1}} & \longrightarrow & \frac{C_n}{dC_{n+1}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_A} & & \downarrow{\scriptstyle d_B} & & \downarrow{\scriptstyle d_C} & & \\ 0 & \longrightarrow & Z_{n-1}(A) & \longrightarrow & Z_{n-1}(B) & \longrightarrow & Z_{n-1}(C) & & \end{array}$$

Notice that

$$\operatorname{Ker}\left(\frac{A_n}{dA_{n+1}} \xrightarrow{d} Z_{n-1}(A)\right) = H_n(A)$$

and

$$\operatorname{Coker}\left(\frac{A_n}{dA_{n+1}} \xrightarrow{d} Z_{n-1}(A)\right) = H_{n-1}(A)$$

and the other two columns are similar. By the Snake Lemma again, we have the connecting map:

$$H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C)$$

Putting all these exact sequences together, we get the desired long exact sequence.

The explicit expression for ∂_n follows directly from the Snake Lemma. ∎

Theorem 6.3.3 (Naturality of ∂). Given a morphism between short exact sequences of chain complexes, i.e., a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_\bullet & \xrightarrow{f} & B_\bullet & \xrightarrow{g} & C_\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A'_\bullet & \xrightarrow{f'} & B'_\bullet & \xrightarrow{g'} & C'_\bullet & \longrightarrow & 0 \end{array}$$

then there is a morphism between long exact sequence, i.e., a commutative diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H_n(A) & \xrightarrow{f_*} & H_n(B) & \xrightarrow{g_*} & H_n(C) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \alpha_*} & & \\ \dots & \longrightarrow & H_n(A') & \xrightarrow{f'_*} & H_n(B') & \xrightarrow{g'_*} & H_n(C') & \xrightarrow{\partial'} & H_{n-1}(A') & \longrightarrow & \dots \end{array}$$

Proof. (Rotman 2009, Theorem 6.13). Since H_n is a functor, the leftmost two squares commute. Take $[c] \in H_n(C)$ for some $c \in Z_n(C)$, we need to show that $\alpha_*\partial([c]) = \partial'\gamma_*([c])$.

Let $b \in B_n$ be a lifting of c, i.e., $g(b) = c$. Then $\partial([c]) = [a]$, where $f(a) = d_B(b)$. Therefore, $\alpha_*\partial([c]) = [\alpha(a)]$.

On the other hand, since γ is a chain map, we have $g'\beta(b) = \gamma g(b) = \gamma(c)$. We see that $b' := \beta(b) \in B'_n$ is a lifting of c' because $g'(b') = g'(\beta(b)) = \gamma(g(b)) = \gamma(c) = c'$. Hence $\partial'\gamma_*([c]) = \partial'([\gamma(c)]) = [a']$, where $f'(a') = d_{B'}(b') = d_{B'}(\beta(b))$.

But

$$f'(\alpha(a)) = \beta(f(a)) = \beta(d_B(b)) = d_{B'}(\beta(b)) = f'(a')$$

and f' is injective, so $\alpha(a) = a'$. ∎

Corollary 6.3.4. Let $\mathcal{A}$ be an abelian category. Then homology induces a functor from the category of short exact sequences of chain complexes in $\mathcal{A}$ to the category of long exact sequences in $\mathcal{A}$.

6.4. Resolutions

Definition 6.4.1. Let $\mathcal{A}$ be an abelian category. Let M be an object of $\mathcal{A}$. A **left resolution** of M is a complex $P_\bullet$, where $P_i = 0$ for negative i, with morphism $\varepsilon : P_0 \to M$ such that

$$\dots \to P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact. If each P_i is projective, then we call it a **projective resolution**. If $\mathcal{A}$ is R-**Mod** or **Mod**-R and each P_i is a free module, then we call it a **free resolution**.

In the same way, we define **right resolutions** and **injective resolutions**, only reversing all the arrows.

Proposition 6.4.2. $P_\bullet \to M$ is a resolution if and only if the following chain map $f : P_\bullet \to M[0]$

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{d} & P_2 & \xrightarrow{d} & P_1 & \xrightarrow{d} & P_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & & 0 & & 0 & & \varepsilon & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

is a quasi-isomorphism.

Proof. By definition, $P_\bullet$ is a resolution if and only if

(1) $P_\bullet$ is exact at P_n for $n \geq 1$ and
(2) $M = \mathrm{Coker}\left(P_1 \xrightarrow{d} P_0\right)$.

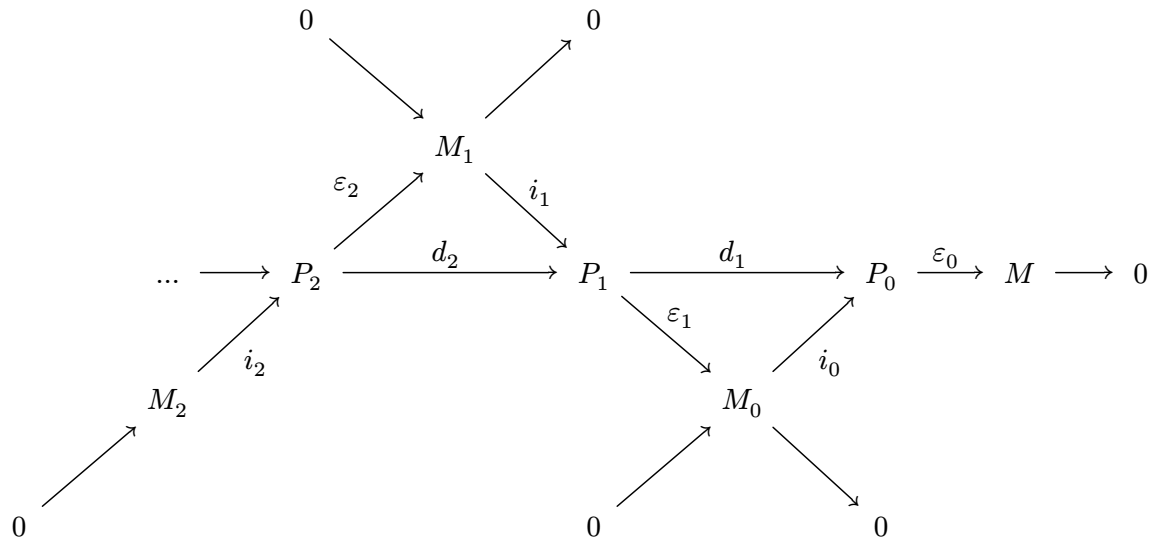
On the other hand, f is quasi-isomorphism if and only if
(3) $H_n(P) \cong H_n(M[0]) \cong 0$ for $n \geq 1$ and
(4) $H_0(P) \cong H_0(M[0]) \cong M$.

(1) is obviously equivalent to (3). (4) is equivalent to $M \cong P_0/\operatorname{Im}(d) = \mathrm{Coker}\left(P_1 \xrightarrow{d} P_0\right)$ and thus equivalent to (2). ∎

By finding a resolution of a potentially "complicated" object M, we can work with a chain complex of "simple" objects, e.g. projective or injective objects.

Lemma 6.4.3. If $\mathcal{A}$ has enough projectives, then every object has a projective resolution. Dually, if $\mathcal{A}$ has enough injectives, then every object has an injective resolution.

Proof. (Monnet and Kremnitzer 2021, Lemma 5.20).



Let $M \in \mathcal{A}$. By definition of having enough projectives, let $\varepsilon_0 : P_0 \to M$ be an epimorphism where P_0 is projective. Let $M_0 := \operatorname{Ker} \varepsilon_0$, and we have short exact sequence

$$0 \to M_0 \to P_0 \to M \to 0.$$

Now we can let $\varepsilon_1 : P_1 \to M_0$ be an epimorphism and $M_1 := \operatorname{Ker} \varepsilon_1$, obtaining the short exact sequence

$$0 \to M_1 \to P_1 \to M_0 \to 0.$$

We define $d_1 = i_0 \varepsilon_1 : P_1 \to P_0$ and then

$$d_1(P_1) = M_0 = \operatorname{Ker} \varepsilon_0.$$

Thus the chain in exact at P_0. The procedure above can be then iterated for any $n \geq 1$ and the resultant chain is infinitely long. ■

Theorem 6.4.4 (Comparison Theorem). In an abelian category $\mathcal{A}$, let $f' : M \to N$. Consider the commutative diagram, where the rows are chain complexes:

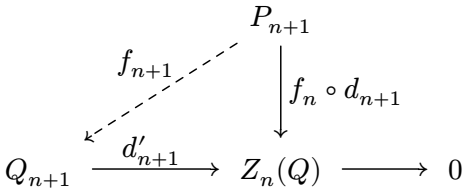
$$\begin{array}{ccccccccccc}
\longrightarrow & P_2 & \xrightarrow{d} & P_1 & \xrightarrow{d} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f'} & & \\
\longrightarrow & Q_2 & \xrightarrow{d'} & Q_1 & \xrightarrow{d'} & Q_0 & \xrightarrow{\eta} & N & \longrightarrow & 0
\end{array}$$

Assume that P_n is projective for all $n \geq 0$ and that $\eta : Q_\bullet \to N$ is a resolution (i.e., the bottom row is exact), then there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ lifting f' (i.e., $f_\bullet$ makes the above diagram commutative). Further, $f_\bullet$ is unique up to a chain homotopy equivalence.

Proof. (Weibel 1994, Comparison Theorem 2.26), (Rotman 2009, Theorem 6.16). Set $f_{-1} = f'$. By induction, suppose that f_n has been constructed. Note that for any $a \in P_{n+1}$, we have

$$d'_n \circ f_n \circ d_{n+1}(a) = f_{n-1} \circ d_n \circ d_{n+1}(a) = 0,$$

therefore $f_n \circ d_{n+1} : P_{n+1} \to Q_n$ lands in $Z_n(Q)$. On the other hand, due to the exactness of $Q_\bullet$, the map $d'_{n+1} : Q_{n+1} \to Z_n(Q)$ is an epimorphism. So we have the following:



where since P_{n+1} is an projective object, the morphism $f_{n+1} : P_{n+1} \to Q_{n+1}$ exists such that the diagram commutes, i.e. $d'_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}$.

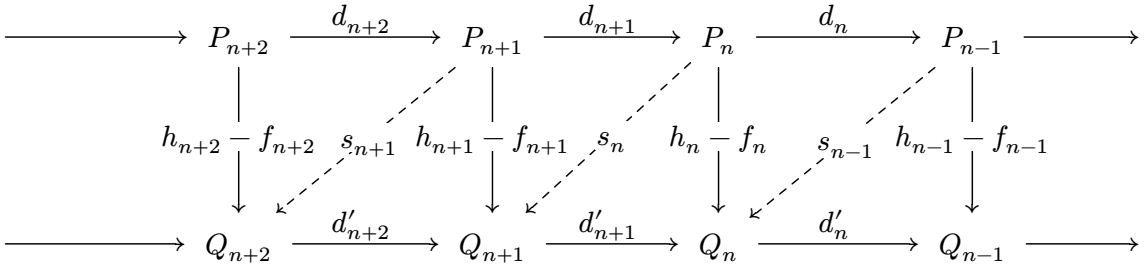
For the uniqueness, let $h : P_\bullet \to Q_\bullet$ be another chain map lifting f'. We want to construct homotopy s with terms $s_n : P_n \to Q_{n+1}$ such that

$$h_n - f_n = d'_{n+1} s_n + s_{n-1} d_n$$

for all $n \geq -1$.

For the base case, set $f_{-1} = h_{-1} = f'$, $d_0 = \varepsilon$, $d_{-1} = 0$, $d'_0 = \eta$, $d_{-1} = 0$. We construct $s_{-2} = s_{-1} = 0$, and the claim is trivially true for $n = -1$.

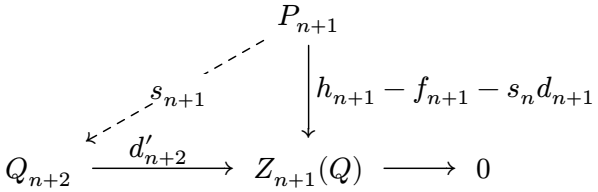
For the induction step, assume we have constructed s_i for $i \leq n$,



(Again, only the solid lines commute.) We want to show the existence of s_{n+1} which satisfies

$$d'_{n+2} s_{n+1} = h_{n+1} - f_{n+1} - s_n d_{n+1}.$$

We claim that $(h_{n+1} - f_{n+1} - s_n d_{n+1})$ sends P_{n+1} to $Z_{n+1}(Q)$. First, notice that this claim would indicate the existence of s_{n+1}, as we would have



where P_{n+1} is projective and $d'_{n+2} : Q_{n+2} \to Z_{n+1}(Q)$ is an epimorphism.

Now

$$\begin{aligned} d'_{n+1}(h_{n+1} - f_{n+1} - s_n d_{n+1}) &= d'_{n+1}(h_{n+1} - f_{n+1}) - d'_{n+1} s_n d_{n+1} \\ &= d'_{n+1}(h_{n+1} - f_{n+1}) - (h_n - f_n - s_{n-1} d_n) d_{n+1} \\ &= d'_{n+1}(h_{n+1} - f_{n+1}) - (h_n - f_n) d_{n+1} \\ &= 0. \end{aligned}$$

Hence $(h_{n+1} - f_{n+1} - s_n d_{n+1})$ sends P_{n+1} to $Z_{n+1}(Q)$. ∎

Lemma 6.4.5 (Horseshoe Lemma). Suppose we have a commutative diagram

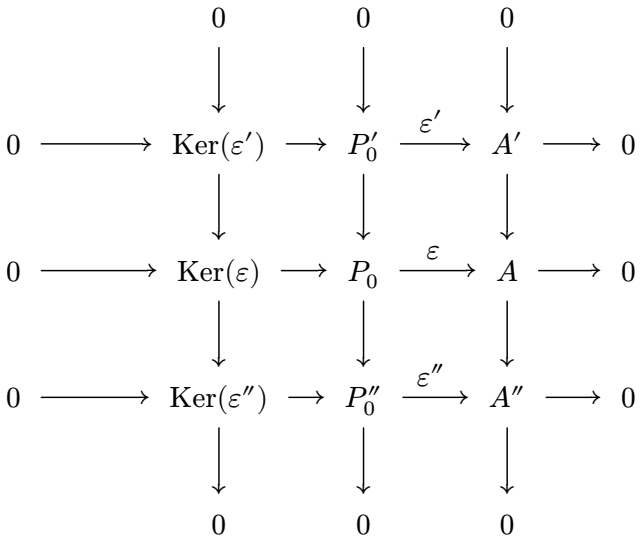
$$\begin{array}{ccccccccccc} & & & & & & & & 0 & & \\ & & & & & & & & \downarrow & & \\ \ldots & \longrightarrow & P'_2 & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \xrightarrow{\varepsilon'} & A' & \longrightarrow & 0 \\ & & & & & & & & \downarrow{\scriptstyle i_A} & & \\ & & & & & & & & A & & \\ & & & & & & & & \downarrow{\scriptstyle \pi_A} & & \\ \ldots & \longrightarrow & P''_2 & \longrightarrow & P''_1 & \longrightarrow & P''_0 & \xrightarrow{\varepsilon''} & A'' & \longrightarrow & 0 \\ & & & & & & & & \downarrow & & \\ & & & & & & & & 0 & & \end{array}$$

where the column is exact and the rows are projective resolutions. Set $P_n = P'_n \oplus P''_n$. Then the P_n assemble to form a projective resolution P of A, and the right-hand column lifts to an exact sequence of complexes

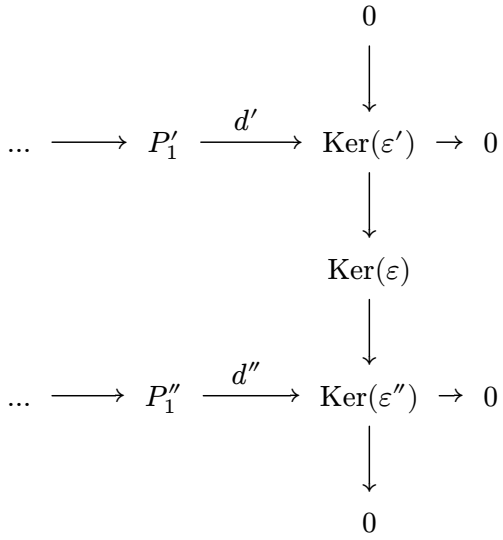
$$0 \to P'_\bullet \xrightarrow{i_\bullet} P_\bullet \xrightarrow{\pi_\bullet} P''_\bullet \to 0,$$

where $i_n : P'_n \to P_n$ and $\pi_n : P_n \to P''_n$ are the natural inclusion and projection, respectively.

Proof. (Weibel 1994, Horseshoe Lemma 2.2.8). Since P''_0 is projective and $\pi_A : A \to A''$ is an epimorphism, we can lift $\varepsilon'' : P''_0 \to A''$ to a map $\widetilde{\varepsilon''} : P''_0 \to A$. The direct sum of $\widetilde{\varepsilon''}$ and $i_A \varepsilon' : P'_0 \to A$ gives a map $\varepsilon : P_0 \to A$. Then the diagram below commutes:



where the right two columns are short exact sequences, and the Snake Lemma 3.7.12 shows that the left column is exact and that $\mathrm{Coker}(\varepsilon) = 0$, so that P_0 maps onto A. This finishes the initial step and brings us to the situation



The filling in of the "horseshoe" now proceeds by induction. ■

7. Derived Functors

7.1. Homological δ-functors

(Weibel 1994, Section 2.1). The next two definitions are stated separately for clarity.

Definition 7.1.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A **homological δ-functor** T from $\mathcal{A}$ to $\mathcal{B}$ is a collection of additive functors $\{T_n : \mathcal{A} \to \mathcal{B}\}_{n \geq 0}$ such that

(1) (Existence of δ). For each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, there exist morphisms $\delta_n : T_n(C) \to T_{n-1}(A)$ for $n \geq 1$ such that

$$\begin{aligned} \ldots \to T_{n+1}(C) \xrightarrow{\delta} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta} T_{n-1}(A) \to \ldots \\ \to T_1(C) \xrightarrow{\delta} T_0(A) \to T_0(B) \to T_0(C) \to 0 \end{aligned}$$

is a long exact sequence in $\mathcal{B}$. In particular, T_0 is right exact;

(2) (Naturality of δ). For each morphism of short exact sequences from $0 \to A' \to B' \to C' \to 0$ to $0 \to A \to B \to C \to 0$, the δ's above give a commutative diagram

$$\begin{array}{ccc} T_n(C') & \xrightarrow{\delta} & T_{n-1}(A') \\ \downarrow & & \downarrow \\ T_n(C) & \xrightarrow{\delta} & T_{n-1}(A) \end{array}$$

Definition 7.1.2. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A **cohomological δ-functor** T from $\mathcal{A}$ to $\mathcal{B}$ is a collection of additive functors $\{T^n : \mathcal{A} \to \mathcal{B}\}_{n \geq 0}$ such that

(1) (Existence of δ). For each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, there exist morphisms $\delta^n : T^n(C) \to T^{n+1}(A)$ for $n \geq 0$ such that

$$\begin{aligned} 0 \to T^0(A) \to T^0(B) \to T^0(C) \xrightarrow{\delta} T^1(A) \to \ldots \\ \to T^{n-1}(C) \xrightarrow{\delta} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta} T^{n+1}(A) \to \ldots \end{aligned}$$

is a long exact sequence in $\mathcal{B}$. In particular, T^0 is left exact;

(2) (Naturality of δ). For each morphism of short exact sequences from $0 \to A' \to B' \to C' \to 0$ to $0 \to A \to B \to C \to 0$, the δ's above give a commutative diagram

$$\begin{array}{ccc} T^n(C') & \xrightarrow{\delta} & T^{n+1}(A') \\ \downarrow & & \downarrow \\ T^n(C) & \xrightarrow{\delta} & T^{n+1}(A) \end{array}$$

Example 7.1.3. Homology gives a homological δ-functor

$$\{H_n : \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{A}\}_{n\geq 0},$$

where $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ is the (full) subcategory of $\mathrm{Ch}(\mathcal{A})$ whose objects are chain complexes $C_\bullet$ such that $C_n = 0$ for all $n < 0$.

Similarly, cohomology gives a cohomological δ-functor

$$\{H^n : \mathrm{Ch}^{\geq 0}(\mathcal{A}) \to \mathcal{A}\}_{n\geq 0},$$

where $\mathrm{Ch}^{\geq 0}(\mathcal{A})$ is defined similarly.

Example 7.1.4. If p is an integer, the collection $\{T_n : \mathbf{Ab} \to \mathbf{Ab}\}_{n\geq 0}$ of functors defined by

$$T_n(A) = \begin{cases} A/pA & n = 0 \\ {}_pA := \{a \in A : pa = 0\} & n = 1 \\ 0 & n \geq 2 \end{cases}$$

form a homological δ-functor (or a cohomological δ-functor with $T^0 = T_1$ and $T^1 = T_0$).

Proof. Apply the Snake Lemma 3.7.12 to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where $A \xrightarrow{p} A$ is the map of multiplication by p and so on, so that we get the exact sequence

$$0 \to {}_pA \to {}_pB \to {}_pC \xrightarrow{\delta} A/pA \to B/pB \to C/pC \to 0.$$

■

Definition 7.1.5. A **morphism** $f : S \to T$ of homological (resp. cohomological) δ-functors is a collection of natural transformations $\{f_n : S_n \to T_n\}_{n\geq 0}$ (resp. $\{f^n : S^n \to T^n\}_{n\geq 0}$) which commutes with δ.

Remark 7.1.6. This definition is equivalent to saying that there is a commutative "ladder diagram" connecting the long exact sequences for S and T associated to any short exact sequence in $\mathcal{A}$.

Definition 7.1.7. A homological δ-functor $T = \{T_n\}$ is **universal** if, given any other homological δ-functor $S = \{S_n\}$ and a natural transformation $f_0 : S_0 \to T_0$, there exists a unique morphism

$$f = \{f_n : S_n \to T_n\}_{n \geq 0} : S \to T$$

extending f_0.

A **universal** cohomological δ-functor T is similarly defined.

Example 7.1.8. If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor, then $T_0 = F$ and $T_n = 0$ for $n \neq 0$ defines a universal homological δ-functor $T : \mathcal{A} \to \mathcal{B}$.

7.2. Derived Functors

The main object of this section is to show that in an abelian category with enough projectives, left derived functors, defined as follows, are homological δ-functors.

Definition 7.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. Assume that $\mathcal{A}$ has enough projectives. For any $A \in \mathcal{A}$, pick a projective resolution $P_\bullet \to A$ by Lemma 6.4.3. Then L_iF given by

$$L_iF(A) := H_i(F(P))$$

is called the i-**th left derived functor**.

Remark 7.2.2. (Rotman 2009, p. 344). To elaborate, given $F : \mathcal{A} \to \mathcal{B}$ and $A \in \mathcal{A}$, to calculate $L_iF(A)$ we need the following steps:

(1) Find a projective resolution of A in $\mathcal{A}$:

$$\ldots \to P_2 \to P_1 \to P_0 \to A \to 0;$$

(2) Delete A to form the **deleted projective resolution**, i.e., the chain complex

$$\ldots \to P_2 \to P_1 \to P_0 \to 0,$$

(which is not exact at P_0 unless $A = 0$);

(3) Apply F to form a chain complex in $\mathcal{B}$:

$$\ldots \to F(P_2) \to F(P_1) \to F(P_0) \to 0;$$

(4) Calculate the i-th homology $H_i(F(P))$ of this chain complex.

In fact, our definition of the "functor" L_iF is still incomplete as we have not defined how it maps the morphisms in $\mathcal{A}$. However, we first need to show that for any object $A \in \mathcal{A}$, our definition of $L_iF(A)$ is independent of the choice of projective resolution $P_\bullet \to A$. The following implies the case when $i = 0$.

Lemma 7.2.3. $L_0F(A) \cong F(A)$.

Proof. Consider the projective resolution of A:

$$...P_1 \xrightarrow{d_1} P_0 \to A \to 0$$

By definition, $L_0F(A) = H_0(F(P)) \cong \text{Coker}(F(d_1))$. Since F is right exact, it preserves cokernels, so $\text{Coker}(F(d_1)) \cong F(\text{Coker}(d_1)) = F(A)$. ∎

Lemma 7.2.4. Let $\mathcal{A}, \mathcal{B}, F, A$ be defined as in Definition 7.2.1. If $P_\bullet \to A$ and $Q_\bullet \to A$ are two projective resolutions, then there is a canonical isomorphism

$$H_i(F(P)) \cong H_i(F(Q))$$

Proof. By the Comparison Theorem 6.4.4, there is a chain map $f : P_\bullet \to Q_\bullet$ lifting the identity $\text{id}_A : A \to A$, which gives $f_* : H_iF(P) \to H_iF(Q)$. Notice that any other lift $f' : P_\bullet \to Q_\bullet$ is chain homotopic to f so $f_* = f'_*$, so f_* is canonical. We can also lift id_A to a map $g : Q_\bullet \to P_\bullet$ and get $g_* : H_iF(Q) \to H_iF(P)$.

Notice $g \circ f : P_\bullet \to P_\bullet$ and $\text{id}_P : P_\bullet \to P_\bullet$ are both chain maps lifting id_A, and by the Comparison Theorem 6.4.4 they are chain homotopic. Therefore $g_* \circ f_* = (g \circ f)_* = (\text{id}_P)_*$. Similarly, $f_* \circ g_* = (\text{id}_Q)_*$, which gives an isomorphism $H_i(F(P)) \cong H_i(F(Q))$. ∎

Corollary 7.2.5. If A is projective, then $L_iF(A) = 0$ for $i \neq 0$.

Proof. Simply notice that $... \to 0 \to A \to A \to 0$ is a projective resolution of A. ∎

Now we complete the definition of L_iF and prove that it is indeed a functor.

Lemma 7.2.6. If $f : A' \to A$ a morphism in $\mathcal{A}$, then there is a natural map

$$L_iF(f) : L_iF(A') \to L_iF(A)$$

Proof. Let $P'_\bullet \to A'$ and $P_\bullet \to A$ be projective resolutions. By the Comparison Theorem 6.4.4, f lifts to a chain map $\tilde{f} : P'_\bullet \to P_\bullet$, which gives a map $\tilde{f}_* : H_iF(P') \to H_iF(P)$. As any other lift is chain homotopic to $\tilde{f}$, the map $\tilde{f}_*$ is independent of the lift. ∎

Proposition 7.2.7. $L_iF : \mathcal{A} \to \mathcal{B}$ is an additive functor.

Proof. Let $A \in \mathcal{A}$ and $P_\bullet \to A$ be a projective resolution. The chain map id_P lifts id_A, so $L_iF(\text{id}_A) = \text{id}_{L_iF(A)}$. Given $A' \xrightarrow{f} A \xrightarrow{g} A''$ in $\mathcal{A}$ and projective resolutions $P'_\bullet \to A'$, $P_\bullet \to A$, and $P''_\bullet \to A''$, we

obtain lifts $\tilde{f} : P'_\bullet \to P_\bullet$ and $\tilde{g} : P_\bullet \to P''_\bullet$. Then the composition $\tilde{g} \circ \tilde{f} : P' \to P''$ is a lift of $g \circ f$, so $g_* \circ f_* = (gf)_* : H_i F(P') \to H_i F(P'')$. Therefore, $L_i F$ is a functor.

If chain maps $\tilde{f}_1, \tilde{f}_2 : P'_\bullet \to P_\bullet$ lift $f_1, f_2 : A' \to A$, then the chain map $\tilde{f}_1 + \tilde{f}_2$ lifts $f_1 + f_2$, so $f_{1*} + f_{2*} = (f_1 + f_2)_* : H_i F(P') \to H_i F(P)$. Therefore, $L_i F$ is an additive functor. ■

Theorem 7.2.8. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor, then $\{L_i F\}_{i \geq 0}$ forms a universal homological δ-functor.

Proof. (Weibel 1994, Theorem 2.4.6 and Theorem 2.4.7). First notice that $L_0 F = F$ is right exact. Given a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

and projective resolutions $P'_\bullet \to A'$ and $P''_\bullet \to A''$, by the Horseshoe Lemma 6.4.5, there is a projective resolution $P_\bullet \to A$ such that $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ is a short exact sequence of chain complexes and for each n, $0 \to P'_n \to P_n \to P''_n \to 0$ is split. Since F is additive, by Lemma 3.1.12, F preserves biproducts and thus preserves split exact sequences, so

$$0 \to F(P'_n) \to F(P_n) \to F(P''_n) \to 0$$

is split exact in $\mathcal{B}$. (Notice that F is not necessarily an exact functor, so $0 \to P'_n \to P_n \to P''_n \to 0$ being split is crucial.) Hence

$$0 \to F(P'_\bullet) \to F(P_\bullet) \to F(P''_\bullet) \to 0$$

is a short exact sequence of chain complexes. Hence applying homology gives the connecting homomorphisms and a long exact sequence

$$\ldots \to L_{n+1} F(A'') \xrightarrow{\delta} L_n F(A') \to L_n F(A) \to L_n F(A'') \xrightarrow{\delta} L_{n-1} F(A') \to \ldots$$

by Theorem 6.3.2.

We omit the proofs that δ's are natural and that $\{L_i F\}_{i \geq 0}$ is universal. ■

Definition 7.2.9. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that $\mathcal{A}$ has enough injectives and for any $A \in \mathcal{A}$ we have an injective resolution $A \to I^\bullet$. Then the **i-th right derived functor** $R^i F$ is defined as

$$R^i F(A) := H^i(F(I^\bullet))$$

Note 7.2.10. $R^i F(A) = (L_i F^{\mathrm{op}})^{\mathrm{op}}(A)$.

Corollary 7.2.11. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor, then $\{R^i F\}_{i \geq 0}$ forms a universal cohomological δ-functor.

8. Balancing Ext and Tor

8.1. Defining Ext and Tor

Definition 8.1.1. Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathcal{A}$ and let $B \to I_\bullet$ be an injective resolution. Recall that $\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathbf{Ab}$ is left exact by Lemma 3.5.14. If $\mathcal{A}$ has enough injectives, we define the right derived functor $\mathrm{Ext}^i_{\mathcal{A}}(A, -)$ of $\mathrm{Hom}_{\mathcal{A}}(A, -)$ as

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) = \mathrm{Ext}^i_{\mathcal{A}}(A, -)(B) := R^i \mathrm{Hom}_{\mathcal{A}}(A, -)(B) = H^i(\mathrm{Hom}_{\mathcal{A}}(A, I^\bullet)).$$

In particular, $\mathrm{Ext}^0_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B)$.

Notice that the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, B) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is also left exact by Corollary 3.5.16. Assume that $\mathcal{A}$ has enough projectives, so $\mathcal{A}^{\mathrm{op}}$ has enough injectives. Let $P_\bullet \to A$ be an projective resolution in $\mathcal{A}$, which can be seen as an injective resolution in $\mathcal{A}^{\mathrm{op}}$. We can thus define another right derived functor $\mathrm{Ext}^i_{\mathcal{A}}(-, B)$, given by

$$\mathrm{Ext}^i_{\mathcal{A}}(-, B)(A) := R^i \mathrm{Hom}_{\mathcal{A}}(-, B)(A) = H^i(\mathrm{Hom}_{\mathcal{A}}(P_\bullet, B)).$$

The above two constructions are in fact isomorphic, i.e., $\mathrm{Ext}^i_{\mathcal{A}}(A, -)(B) \cong \mathrm{Ext}^i_{\mathcal{A}}(-, B)(A)$, or

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) := R^i \mathrm{Hom}_{\mathcal{A}}(A, -)(B) \cong R^i \mathrm{Hom}_{\mathcal{A}}(-, B)(A).$$

This isomorphism is called the **balancing of** Ext. Before proving the balancing of Ext, we present some properties of Ext that it gives.

Proposition 8.1.2. Let $0 \to K \to L \to M \to 0$ be a short exact sequence in $\mathcal{A}$ and let $A, B \in \mathcal{A}$. Then we have the induced long exact sequences

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{A}}(A, K) \to \mathrm{Hom}_{\mathcal{A}}(A, L) \to \mathrm{Hom}_{\mathcal{A}}(A, M) \to \\ \mathrm{Ext}^1_{\mathcal{A}}(A, K) \to \mathrm{Ext}^1_{\mathcal{A}}(A, L) \to \mathrm{Ext}^1_{\mathcal{A}}(A, M) \to \dots \end{aligned}$$

and

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{A}}(M, B) \to \mathrm{Hom}_{\mathcal{A}}(L, B) \to \mathrm{Hom}_{\mathcal{A}}(K, B) \to \\ \mathrm{Ext}^1_{\mathcal{A}}(M, B) \to \mathrm{Ext}^1_{\mathcal{A}}(L, B) \to \mathrm{Ext}^1_{\mathcal{A}}(K, B) \to \dots \end{aligned}$$

Proof. Simply notice that $\{\mathrm{Ext}^i_{\mathcal{A}}(A, -)\}_{i \geq 0}$ and $\{\mathrm{Ext}^i_{\mathcal{A}}(-, B)\}_{i \geq 0}$ form two cohomological δ-functors. ∎

Proposition 8.1.3. The followings are equivalent:
(1) B is injective;
(2) $\mathrm{Hom}_{\mathcal{A}}(-, B)$ is exact;
(3) $\mathrm{Ext}^i_{\mathcal{A}}(A, B) = 0$ for $i \neq 0$ and all A;
(4) $\mathrm{Ext}^1_{\mathcal{A}}(A, B) = 0$ for all A.

Proof. (1) $\Leftrightarrow$ (2) by the definition of injective objects.

(1) $\Rightarrow$ (3) by applying the dual of Corollary 7.2.5 to $\mathrm{Ext}^i_{\mathcal{A}}(A, -)$.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (2). Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence in $\mathcal{A}$, which induces the long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A', B) \to \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{A}}(A'', B) \to \mathrm{Ext}^1_{\mathcal{A}}(A', B) \to \dots$$

Since $\mathrm{Ext}^1_{\mathcal{A}}(A', B) = 0$ by assumption, $\mathrm{Hom}_{\mathcal{A}}(-, B)$ is an exact functor. ■

Proposition 8.1.4. The followings are equivalent:
(1) A is projective;
(2) $\mathrm{Hom}_{\mathcal{A}}(A, -)$ is exact;
(3) $\mathrm{Ext}^i_{\mathcal{A}}(A, B) = 0$ for $i \neq 0$ and all B;
(4) $\mathrm{Ext}^1_{\mathcal{A}}(A, B) = 0$ for all B.

Example 8.1.5. Let $m, n \in \mathbb{Z}$. Let us calculate $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n)$ in two different ways.

We may use the injective resolution of $\mathbb{Z}/n$:

$$0 \to \mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z} \xrightarrow{n} \mathbb{Q}/\mathbb{Z} \to 0.$$

Now delete $\mathbb{Z}/n$, apply $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m, -)$, and use $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/m$, we get

$$0 \to \mathbb{Z}/m \xrightarrow{n} \mathbb{Z}/m \to 0.$$

Calculating the first cohomology of this sequence reveals that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = H^1 = \mathrm{Coker}\left(\mathbb{Z}/m \xrightarrow{n} \mathbb{Z}/m\right) \cong \mathbb{Z}/\gcd(m, n)$.

On the other hand, we may invoke the balancing of Ext and use the projective resolution of $\mathbb{Z}/m$:

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0.$$

Now delete $\mathbb{Z}/m$, apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/n)$ (which is a contravariant functor), and use $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n) \cong \mathbb{Z}/n$, we get

$$0 \to \mathbb{Z}/n \xrightarrow{m} \mathbb{Z}/n \to 0.$$

Again the first cohomology of the sequence gives $\mathbb{Z}/\gcd(m, n)$.

Definition 8.1.6. Let R be a ring and B be a left R-module. Since $(- \otimes_R B) : \mathbf{Mod}\text{-}R \to \mathbf{Ab}$ is right exact by Corollary 4.5.4 and R-$\mathbf{Mod}$ has enough projectives, we can define the left derived functor $\mathrm{Tor}^R_i(-, B)$:

$$\mathrm{Tor}^R_i(A, B) = \mathrm{Tor}^R_i(-, B)(A) := L_i(- \otimes_R B)(A).$$

Similarly, let A be a right R-module, and $(A \otimes_R -) : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is right exact by Theorem 4.5.5. We can thus define the left derived functor $\mathrm{Tor}_i^R(A, -)$:

$$\mathrm{Tor}_i^R(A, -)(B) := L_i(A \otimes_R -)(B).$$

The two constructions are again isomorphic, i.e.,

$$\mathrm{Tor}_i^R(A, B) := L_i(- \otimes_R B)(A) \cong L_i(A \otimes_R -)(B).$$

This isomorphism is called **the balancing of** Tor, which gives the following property.

Proposition 8.1.7. Let $0 \to K \to L \to M \to 0$ be a short exact sequence in $\mathbf{Mod}\text{-}R$ and let $B \in R\text{-}\mathbf{Mod}$. Then we have the induced long exact sequence

$$\ldots \to \mathrm{Tor}_1^R(K, B) \to \mathrm{Tor}_1^R(L, B) \to \mathrm{Tor}_1^R(M, B) \to K \otimes_R B \to L \otimes_R B \to M \otimes_R B \to 0.$$

If $0 \to K \to L \to M \to 0$ is instead a short exact sequence in $R\text{-}\mathbf{Mod}$ and let $A \in \mathbf{Mod}\text{-}R$, then we have the induced long exact sequence

$$\ldots \to \mathrm{Tor}_1^R(A, K) \to \mathrm{Tor}_1^R(A, L) \to \mathrm{Tor}_1^R(A, M) \to A \otimes_R K \to A \otimes_R L \to A \otimes_R M \to 0.$$

In order to prove the balancing of Ext and Tor, we need two new tools: mapping cones and double complexes, introduced in the following sections.

8.2. Mapping Cones

Definition 8.2.1. Let $f : B_\bullet \to C_\bullet$ be a chain map. Define the **mapping cone** of f as the chain complex $\mathrm{cone}(f)_\bullet$, given by

$$\mathrm{cone}(f)_n = B_{n-1} \oplus C_n$$

with differential[4]

$$d(b, c) = (-d(b), d(c) - f(b))$$

for $b \in B_{n-1}$ and $c \in C_n$. We could also write the differential in the form of a matrix:

$$\begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix} : \begin{pmatrix} B_{n-1} \\ C_n \end{pmatrix} \to \begin{pmatrix} B_{n-2} \\ C_{n-1} \end{pmatrix}$$

Dually, let $g : B^\bullet \to C^\bullet$ be a cochain map, then the mapping cone of g is the cochain complex $\mathrm{cone}(g)^\bullet$ given by

$$\mathrm{cone}(g)^n = B^{n+1} \oplus C^n$$

with differential

$$d(b, c) = (-d(b), d(c) - f(b))$$

for $b \in B^{n+1}$ and $c \in C^n$.

[4] In Example 8.3.7 there is an explanation for this definition.

Lemma 8.2.2. Let $f : B_\bullet \to C_\bullet$ be a chain map. Then there is a long exact sequence in homology

$$\ldots \to H_{n+1}(\mathrm{cone}(f)) \to H_n(B) \xrightarrow{\partial} H_n(C) \to H_n(\mathrm{cone}(f)) \to \ldots$$

where the connecting morphism $\partial = f_*$.

Proof. There is a short exact sequence of chain complexes:

$$0 \to C \xrightarrow{i} \mathrm{cone}(f) \xrightarrow{\pi} B[-1] \to 0,$$

where $i : c \mapsto (0, c)$ and $\pi : (b, c) \mapsto -b$. Notice that $H_{n+1}(B[-1]) = H_n(B)$, so we get the corresponding long exact sequence in homology as above by Theorem 6.3.2.

Further, we have $\partial = i^{-1} d_{\mathrm{cone}(f)} \pi^{-1}$ by Theorem 6.3.2. Let $b \in B_n$ be a cycle. We can lift it to $(-b, 0)$ in $\mathrm{cone}(f)$. Apply the differential of $\mathrm{cone}(f)$ to get $d_{\mathrm{cone}(f)}(-b, 0) = (d(b), f(b)) = (0, f(b))$. Thus $\partial[b] = [f(b)] = f_*[b]$. ∎

The following is the main function of the mapping cone.

Corollary 8.2.3. A chain map $f : B_\bullet \to C_\bullet$ is a quasi-isomorphism if and only if $\mathrm{cone}(f)$ is acyclic.

Proof. "⇒". If f is a quasi-isomorphism, then $f_* : H_n(B) \to H_n(C)$ is an isomorphism for all n. Then we have an exact sequence

$$H_n(B) \xrightarrow{f_*} H_n(C) \xrightarrow{i_*} H_n(\mathrm{cone}(f)) \xrightarrow{\pi_*} H_{n-1}(B) \xrightarrow{f_*} H_{n-1}(C).$$

By exactness at $H_n(C)$, we have that $\mathrm{Ker}(i_*) = \mathrm{Im}(f_*) = H_n(C)$. So $i_* = 0$ and $\mathrm{Im}(i_*) = 0$. By exactness at $H_{n-1}(B)$, we have that $\mathrm{Im}(\pi_*) = \mathrm{Ker}(f_*) = 0$, so $\pi_* = 0$ and $\mathrm{Ker}(\pi_*) = H_n(\mathrm{cone}(f))$.

By exactness at $H_n(\mathrm{cone}(f))$, we have

$$0 = \mathrm{Im}(i_*) = \mathrm{Ker}(\pi_*) = H_n(\mathrm{cone}(f)),$$

so $\mathrm{cone}(f)$ is acyclic.

"⇐". If cone (f) is acyclic, then $H_n(\mathrm{cone}(f)) = 0$ and we have an exact sequence

$$0 \to H_n(B) \xrightarrow{f_*} H_n(\mathrm{cone}(f)) \to 0,$$

which indicates that f_* is an isomorphism. ∎

Remark 8.2.4. The same result can be obtained for cochain maps.

There is a similar construction called the mapping cylinder, although we do not use it in these notes.

Definition 8.2.5. The **mapping cylinder** of a chain map $f : B_\bullet \to C_\bullet$ is defined as the chain complex $\mathrm{cyl}(f)_n = B_n \oplus B_{n-1} \oplus C_n$. The differential can be represented by the matrix

$$\begin{pmatrix} d_B & \mathrm{id}_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{pmatrix}.$$

Remark 8.2.6. The reader is directed to (Weibel 1994, Section 1.5) for some topological remarks on mapping cones and mapping cylinders.

8.3. Double and Total Complexes

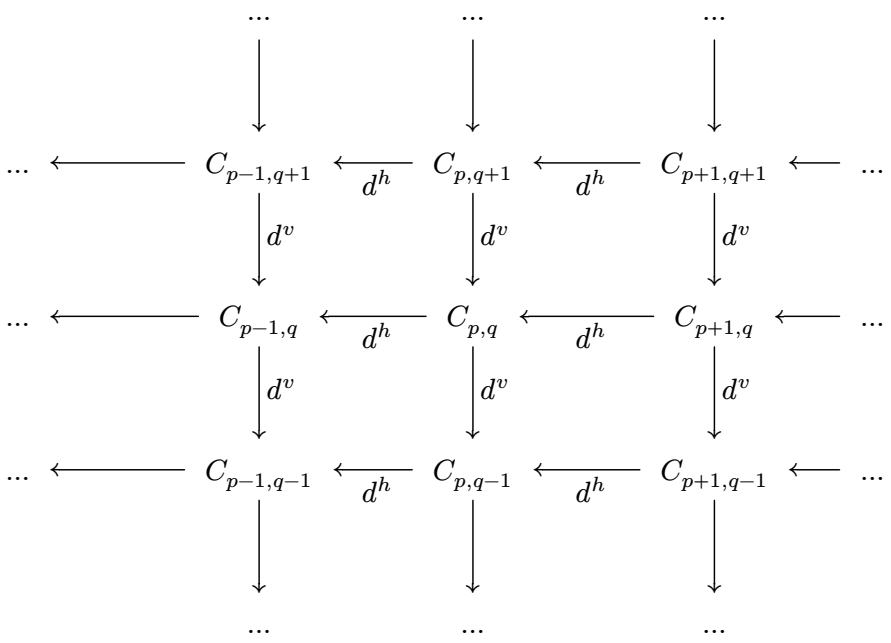
Recall that if $\mathcal{A}$ is an abelian category, $\mathrm{Ch}(\mathcal{A})$ is also an abelian category. Then to define a "two-dimensional" complex, one may be tempted to consider the category $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$. However, what we define next is slightly different from that.

Definition 8.3.1. A **double complex** (or **bicomplex**) $C = C_{\bullet\bullet}$ in an abelian category $\mathcal{A}$ is a family $\{C_{p,q}\}$ of objects in $\mathcal{A}$ with maps $d^h_{p,q} : C_{p,q} \to C_{p-1,q}$ and $d^v_{p,q} : C_{p,q} \to C_{p,q-1}$ such that

$$\left(d^h\right)^2 = \left(d^v\right)^2 = 0, \quad d^v d^h + d^h d^v = 0.$$

The **total degree** of a term $C_{p,q}$ is defined as $p + q$.

In other words, a double complex is an infinite two-dimensional grid of objects where each row (resp. each column) is a chain complex, and the horizontal and vertical differentials *anticommute*. A diagram for a double complex is shown as below; this is not a commutative (but an anticommutative) diagram.



Remark 8.3.2. Because the differentials anticommute, d^v cannot be seen as chain maps between rows. We need to replace $d^v_{p,q}$ by $f_{p,q} := (-1)^p d^v_{p,q}$ (so that the signs alternate for adjacent columns) to make the squares commute. For example, the following is a commutative diagram:

$$\begin{array}{ccccccc}
C_{0,1} & \xleftarrow{d^h} & C_{1,1} & \xleftarrow{d^h} & C_{2,1} & \xleftarrow{d^h} & C_{3,1} \\
\downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle -d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle -d^v} \\
C_{0,0} & \xleftarrow{d^h} & C_{1,0} & \xleftarrow{d^h} & C_{2,0} & \xleftarrow{d^h} & C_{3,0}
\end{array}$$

Therefore, $f_{\bullet,q} : C_{\bullet,q} \to C_{\bullet,q-1}$ is a chain map between two adjacent rows. This also gives an isomorphism between the category of bicomplexes in $\mathcal{A}$ and $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$.

Definition 8.3.3. Let $C_{\bullet\bullet}$ be a double complex. We say that $C_{\bullet\bullet}$ is an **upper half-plane complex** if there is some q_0 such that $C_{p,q} = 0$ for all $q < q_0$. Similarly, $C_{\bullet\bullet}$ is a **right half-plane complex** if there is some p_0 such that $C_{p,q} = 0$ for all $p < p_0$.

Definition 8.3.4. Given $C = \{C_{p,q}\}$, we can define the **total complex** $\mathrm{Tot}^{\Pi}(C)$, given by

$$\mathrm{Tot}^{\Pi}(C)_n = \prod_{p+q=n} C_{p,q}.$$

That is, the n-th term of $\mathrm{Tot}^{\Pi}(C)$ is the product of all terms in C which has total degree n. When for each n, only finitely many terms in C has total degree n, we also define $\mathrm{Tot}^{\oplus}(C)$, given by

$$\mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

$\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ both have differential

$$d = d^h + d^v.$$

Notation 8.3.5. If C is a double complex, sometimes we write $H_n(C)$ to mean $H_n(\mathrm{Tot}^{\Pi}(C))$ or $H_n(\mathrm{Tot}^{\oplus}(C))$.

Lemma 8.3.6. In a total complex, we have that $d^2 = 0$, so the total complex is indeed a chain complex.

Proof. (Rotman 2009, Lemma 10.5).

$$d^2 = \left(d^h + d^v\right)\left(d^h + d^v\right) = \left(d^h\right)^2 + \left(d^h d^v + d^v d^h\right) + \left(d^v\right)^2 = 0.$$

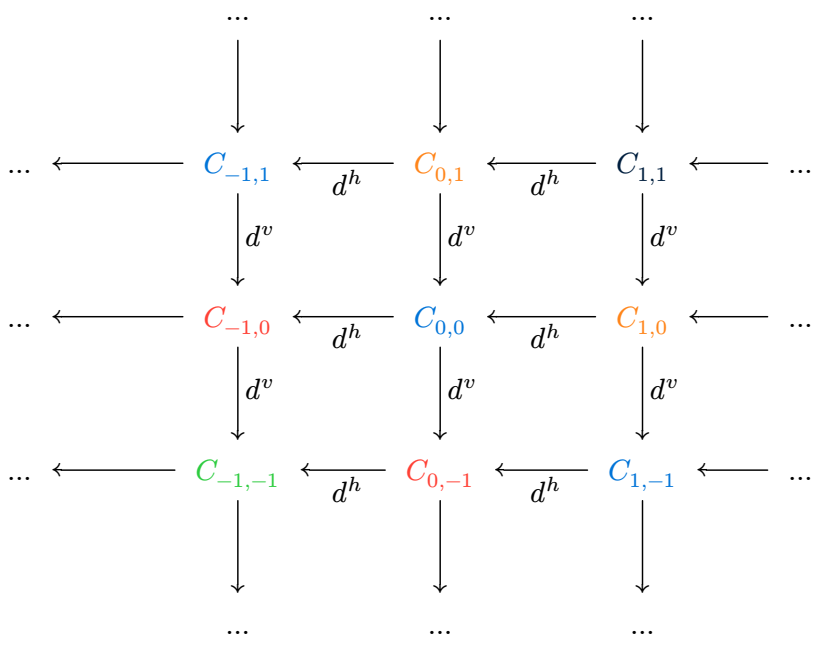
(This is why we have defined double complexes in the anticommuting way.) ∎

The total complex is illustrated by the colours in the following diagram; each "diagonal slice" is given a different colour. For example, $\mathrm{Tot}(C_{\bullet\bullet})_0$ is the product of all the blue terms. This diagram also helps explain how the differential of the total complex works. For example, take

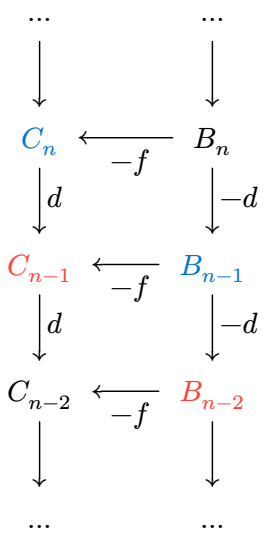
$$c = (..., c_{-1,1}, c_{0,0}, c_{1,-1}, ...) \in \prod_{p\in\mathbb{Z}} C_{-p,p} = \mathrm{Tot}(C)_0.$$

Then

$$d(c) = \left(..., \underbrace{d^v(c_{-1,1}) + d^h(c_{0,0})}_{\in C_{-1,0}}, \underbrace{d^v(c_{0,0}) + d^h(c_{1,-1})}_{\in C_{0,-1}}, ... \right) \in \mathrm{Tot}(C)_{-1}.$$



Example 8.3.7. Let $f_\bullet : B_\bullet \to C_\bullet$ be a chain map, then the following diagram forms a (two-column) double complex. The reader is welcome to verify that the total complex of this double complex is exactly $\mathrm{cone}(f)$, which in particular helps explain why we have defined the differential of $\mathrm{cone}(f)$ in that way.



Lemma 8.3.8 (Acyclic Assembly Lemma). Let $C = \{C_{p,q}\}$ be a double complex. If
(1) C is an upper half-plane complex with exact columns, or
(2) C is a right half-plane complex with exact rows,

then $\mathrm{Tot}^{\Pi}(C)$ is acyclic. If
(3) C is an upper half-plane complex with exact rows, or
(4) C is a right half-lane complex with exact columns,

then $\mathrm{Tot}^{\oplus}(C)$ is acyclic.

Proof. (Weibel 1994, Lemma 2.7.3) explains why proving (1) is sufficient to prove all four conditions, so we work on (1) only. Let C be an upper half-plane bicomplex with exact columns, where we assume $C_{p,q} = 0$ when $q < 0$ (by translating C up or down). It is sufficient to show that

$$H_0(\mathrm{Tot}^{\Pi}(C)) = 0,$$

since by translating C left and right, this will indicate that $H_n(\mathrm{Tot}^{\Pi}(C)) = 0$ for all n.

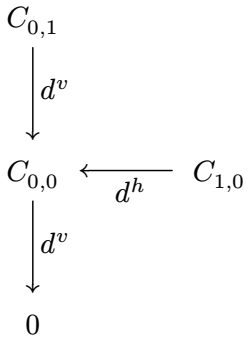
Let

$$c = (..., c_{-2,2}, c_{-1,1}, c_{0,0}) \in \prod C_{-p,p} = \mathrm{Tot}^{\Pi}(C)_0$$

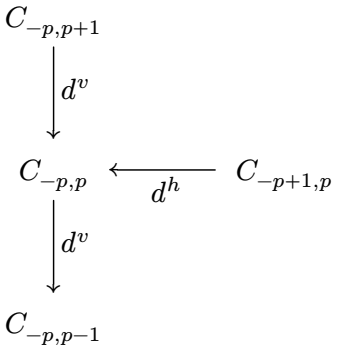
be a 0-cycle, i.e., $d(c) = 0$. We will use induction to find elements $b_{-p,p+1} \in C_{-p,p+1}$ for $p \geq -1$ such that

$$d^v(b_{-p,p+1}) + d^h(b_{-p+1,p}) = c_{-p,p}.$$

For the base case, let $b_{1,0} = 0$ for $p = -1$. Since the 0-th column is exact, there exists $b_{0,1} \in C_{0,1}$ such that $d^v(b_{0,1}) = c_{0,0}$.



By induction, suppose we have found $b_{-p+1,p}$ and want to find $b_{-p,p+1}$.



We compute that

$$\begin{aligned} d^v\big(c_{-p,p} - d^h\big(b_{-p+1,p}\big)\big) &= d^v\big(c_{-p,p}\big) + d^h d^v\big(b_{-p+1,p}\big) \\ &= d^v\big(c_{-p,p}\big) + d^h\big(c_{-p+1,p-1}\big) - d^h d^h\big(b_{-p+2,p-1}\big) \\ &= 0, \end{aligned}$$

where $d^v\big(c_{-p,p}\big) + d^h\big(c_{-p+1,p-1}\big) = 0$ because $d(c) = 0$. Thus

$$c_{-p,p} - d^h\big(b_{-p+1,p}\big) \in \mathrm{Ker}\big(d^v : C_{-p,p} \to C_{-p,p-1}\big) = \mathrm{Im}\big(d^v : C_{-p,p+1} \to C_{-p,p}\big),$$

since the $(-p)$-th column is exact. So there exists $b_{-p,p+1}$ such that

$$d^v\big(b_{-p,p+1}\big) = c_{-p,p} - d^h\big(b_{-p+1,p}\big)$$

as desired. Now assembling all $b_{-p,p+1}$ gives

$$b = \big(\ldots, b_{-1,2}, b_{0,1}, b_{1,0}\big) \in \prod C_{-p,p+1} = \mathrm{Tot}^{\Pi}(C)_1$$

such that $d(b) = c$, which proves that $H_0\big(\mathrm{Tot}^{\Pi}(C)\big) = 0$. ∎

Remark 8.3.9. This lemma is also a consequence of spectral sequences.

A variant of the above lemma is the following, whose proof is similar (Monnet and Kremnitzer 2021, Lemma 8.8).

Lemma 8.3.10. Let C be a double complex such that for every n, there exist only finitely many pairs (p,q) such that $p+q=n$ and $C_{p,q}\neq 0$. If C has exact rows (or if C has exact columns), then $\mathrm{Tot}^{\oplus}(C)$ is acyclic.

8.4. Balancing Tor

Definition 8.4.1. Suppose $\left(P_\bullet, d^{(P)}\right)$ is a chain complex in $\mathbf{Mod}$-R and $\left(Q_\bullet, d^{(Q)}\right)$ is a chain complex in R-$\mathbf{Mod}$. We can form a double complex of abelian groups which we call the **tensor product double complex**, denoted as $P_\bullet \otimes_R Q_\bullet$, where the (p,q) term is $P_p \otimes_R Q_q$ and $d^h_{p,q} = d^{(P)}_p \otimes 1$ and $d^v_{p,q} = (-1)^p \otimes d^{(Q)}_q$. It has the **tensor product total complex**, $\mathrm{Tot}^{\oplus}(P_\bullet \otimes_R Q_\bullet)$.

Lemma 8.4.2. The differentials of $P_\bullet \otimes_R Q_\bullet$ anticommute, so $P_\bullet \otimes_R Q_\bullet$ is a double complex.

Proof. Notice that $\left(d^{(P)}\otimes 1\right)\circ\left(1\otimes d^{(Q)}\right) = d^{(P)}\otimes d^{(Q)} = \left(1\otimes d^{(Q)}\right)\circ\left(d^{(P)}\otimes 1\right)$ by Corollary 4.1.6, and alternating the signs for adjacent columns makes each square anticommute. ∎

Lemma 8.4.3. If P is a projective right R-module, then the functor $(P\otimes_R -) : R\text{-}\mathbf{Mod}\to \mathbf{Ab}$ is exact. If Q is a projective left R-module, then $(-\otimes_R Q) : \mathbf{Mod}\text{-}R \to \mathbf{Ab}$ is exact.[5]

Proof. (Rotman 2009, Proposition 3.46, p. 132). We (very concisely) work on the right R-module case. First notice that $(R\otimes_R -)$ is an isomorphism by Corollary 4.3.2, so the functor $(R\otimes_R -)$ is exact. Then tensor product preserves direct sums by Corollary 4.5.4, so for a family of right R-modules M_i, $((\bigoplus M_i)\otimes_R -)$ is exact, if and only if $\bigoplus(M_i\otimes_R -)$ is exact, if and only if each $(M_i\otimes_R -)$ is exact. Now any free module F, being a direct sum of R's, must have that $(F\otimes_R -)$ is exact. Finally, P is projective, hence P is a direct summand of some free module by Proposition 5.1.2, which indicates that $(P\otimes_R -)$ is also exact. ∎

Theorem 8.4.4 (Balancing of Tor). Let $A\in \mathbf{Mod}\text{-}R$ and $B\in R\text{-}\mathbf{Mod}$. For all n,

$$\mathrm{Tor}^R_n(A,B) := L_n(-\otimes_R B)(A) \cong L_n(A\otimes_R -)(B).$$

Proof. (Weibel 1994, Theorem 2.7.2). (We drop the dots for chain complexes in this proof.) Choose a projective resolution $P_\bullet \xrightarrow{\varepsilon} A$ in $\mathbf{Mod}$-R and a project resolution $Q_\bullet \xrightarrow{\eta} B$ in R-$\mathbf{Mod}$. We can view A, B as chain complexes concentrated in degree 0. Now consider the double complexes $P\otimes_R Q$, $A\otimes_R Q$ and $P\otimes_R B$, and we have *bicomplex morphisms* (where it might be helpful to recall the diagram in Proposition 6.4.2):

[5] This lemma is the same as saying "every projective module is flat", but we have yet to define flat modules. We will revisit this claim in Corollary 10.1.3.

$$\varepsilon \otimes \mathrm{id}_Q : P \otimes_R Q \to A \otimes_R Q$$
$$\mathrm{id}_Q \otimes \eta : P \otimes_R Q \to P \otimes_R B$$

which induce chain maps on the total complexes:

$$f : \mathrm{Tot}^{\oplus}(P \otimes_R Q) \to \mathrm{Tot}^{\oplus}(A \otimes_R Q) = A \otimes_R Q$$
$$g : \mathrm{Tot}^{\oplus}(P \otimes_R Q) \to \mathrm{Tot}^{\oplus}(P \otimes_R B) = P \otimes_R B$$

We claim that f and g are quasi-isomorphisms, which would give isomorphisms on homology and thus prove the result, i.e.

$$H_*(\mathrm{Tot}^{\oplus}(P \otimes_R Q)) \cong H_*(A \otimes_R Q) = L_*(A \otimes_R -)(B)$$
$$H_*(\mathrm{Tot}^{\oplus}(P \otimes_R Q)) \cong H_*(P \otimes_R B) = L_*(- \otimes_R B)(A)$$

Now we form a double complex C, obtained from $P \otimes_R Q$ by adding $A \otimes_R Q$ in the column $p = -1$ using the augmentation $\varepsilon : P_0 \to A$,

$$\begin{array}{ccccccc}
A \otimes Q_2 & \xleftarrow{\varepsilon \otimes 1} & P_0 \otimes Q_2 & \xleftarrow{d_P \otimes 1} & P_1 \otimes Q_2 & \xleftarrow{d_P \otimes 1} & P_2 \otimes Q_2 \\
\downarrow{\scriptstyle -1 \otimes d_Q} & & \downarrow{\scriptstyle 1 \otimes d_Q} & & \downarrow{\scriptstyle -1 \otimes d_Q} & & \downarrow{\scriptstyle 1 \otimes d_Q} \\
A \otimes Q_1 & \xleftarrow{\varepsilon \otimes 1} & P_0 \otimes Q_1 & \xleftarrow{d_P \otimes 1} & P_1 \otimes Q_1 & \xleftarrow{d_P \otimes 1} & P_2 \otimes Q_1 \\
\downarrow{\scriptstyle -1 \otimes d_Q} & & \downarrow{\scriptstyle 1 \otimes d_Q} & & \downarrow{\scriptstyle -1 \otimes d_Q} & & \downarrow{\scriptstyle 1 \otimes d_Q} \\
A \otimes Q_0 & \xleftarrow{\varepsilon \otimes 1} & P_0 \otimes Q_0 & \xleftarrow{d_P \otimes 1} & P_1 \otimes Q_0 & \xleftarrow{d_P \otimes 1} & P_2 \otimes Q_0
\end{array}$$

where $C_{-1,q} = A \otimes Q_q$ and $C_{p,q} = P_p \otimes Q_q$ for any $p, q \geq 0$. Then

$$(\mathrm{Tot}^{\oplus}(C)[-1])_n = \mathrm{Tot}^{\oplus}(C)_{n-1} = \mathrm{Tor}^{\oplus}(P \otimes_R Q)_{n-1} \oplus (A \otimes Q_n)$$

Meanwhile, the mapping cone of $f : \mathrm{Tot}^{\oplus}(P \otimes_R Q) \to A \otimes_R Q$ has

$$\mathrm{cone}(f)_n = \mathrm{Tot}^{\oplus}(P \otimes_R Q)_{n-1} \oplus (A \otimes Q_n).$$

Also

$$d_{\mathrm{cone}(f)} = \left(-\left(d^{(P)} \otimes 1 + (-1)^p \otimes d^{(Q)}\right), 1 \otimes d^{(Q)} - \varepsilon \otimes 1\right) = -d_{\mathrm{Tot}^{\oplus}(C)[-1]},$$

hence $\mathrm{cone}(f) \cong \mathrm{Tot}^{\oplus}(C)[-1]$. To show that f is a quasi-isomorphism, we need to show $\mathrm{cone}(f)$ is acyclic by Corollary 8.2.3. As any Q_p is projective, $\left(- \otimes_R Q_p\right)$ is exact by Lemma 8.4.3. Since $P_\bullet \to A$ is a resolution, every row of C is exact. Since C is upper half-plane, $\mathrm{Tot}^{\oplus}(C)$ is acyclic by Lemma 8.3.8. So f is a quasi-isomorphism.

Similarly, we can show that g is a quasi-isomorphism by forming a double complex C' obtained from adding $B \otimes_R P$ in the row $q = -1$ of $P \otimes_R Q$. ∎

8.5. **Balancing** Ext

Definition 8.5.1. Given a chain complex $\left(P_\bullet, d^{(P)}\right)$ and a cochain complex $\left(I^\bullet, d_{(I)}\right)$, we can form the **Hom double complex**

$$\mathrm{Hom}(P_\bullet, I^\bullet) = \left\{\mathrm{Hom}\left(P_p, I^q\right)\right\}_{p,q}$$

with differentials[6]

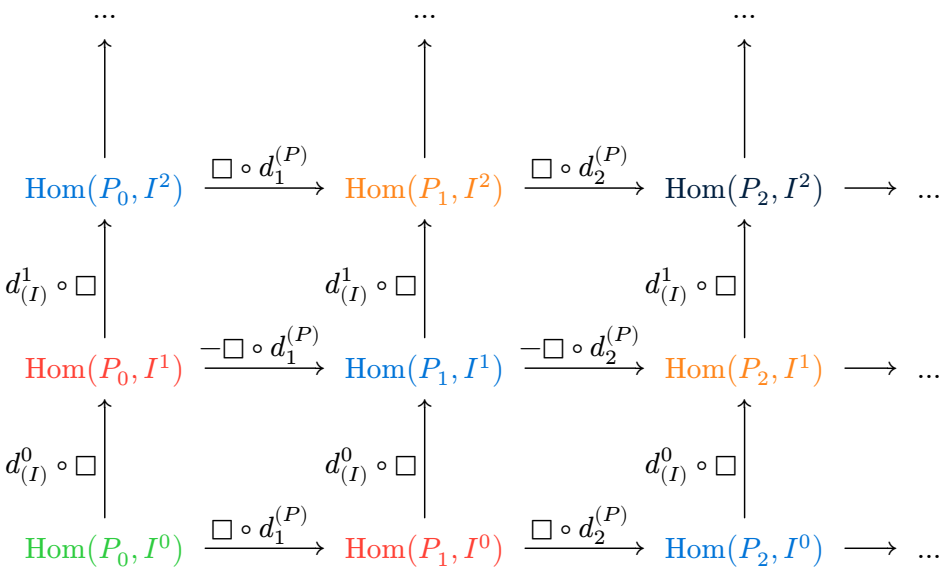
$$d^h_{p,q}(f) = (-1)^q f \circ d^{(P)}_{p+1} \in \mathrm{Hom}\left(P_{p+1}, I^q\right)$$
$$d^v_{p,q}(f) = d^q_{(I)} \circ f \in \mathrm{Hom}\left(P_p, I^{q+1}\right)$$

for $f \in \mathrm{Hom}\left(P_p, I^q\right)$.

Then we define the **Hom cochain complex**[7] as

$$\mathrm{Tot}^{\oplus}(\mathrm{Hom}(P, I))$$

An (anticommutative) diagram for the Hom double complex is as follows. The placeholder in function compositions is written as $\square$ (instead of $-$ as in most parts of these notes) so that it is not confused with the minus sign. Note particularly the signs and indices in the horizontal differentials. Also note that each row and each column is a cochain complex.



Remark 8.5.2. There are a few technicalities to be addressed here. They are not conceptually difficult but can be bewildering when first encountered.

Notice that in our original definition of a double complex, we would draw the arrows pointing downwards and to the left, which we refer to as a **canonical ordering**. However, when we draw the diagram for a Hom double complex, the arrows point upwards and to the right. Thus this

[6] Here we alternate the signs for adjacent rows (instead of adjacent columns, as in the tensor product double complex). This sign convention, following (Monnet and Kremnitzer 2021, p. 76), is different from that in (Weibel 1994, p. 62).

[7] (Weibel 1994, p. 62) writes this as $\mathrm{Tor}^{\Pi}(\mathrm{Hom}(P, I))$, but as we will see in this case any diagonal slice has only finite terms, so their product and direct sum are the same.

is, strictly speaking, neither a upper half-plane complex nor a right half-plane complex, because if we would like to turn the diagram into a canonically ordered one, we would need to reflect it to the "third quadrant". This ordering matters mainly because in this case, it would be more convenient to apply Lemma 8.3.10 instead of Acyclic Assembly Lemma 8.3.8.

Another confusion that can easily arise from a non-canonical ordering is how to form the corresponding total complex. Apart from converting the diagram to a canonically ordered one by reflection, a simple method is to select any object A in the grid and draw a line l connecting the arrowheads of the two arrows departing from A. Then every "diagonal slice", whose direct sum is a term of the total complex, must be parallel to this line l. This is simply because each arrow must point from one diagonal slice to another. For example, each diagonal slice of the Hom double complex has a distinct colour in the above diagram, and hence we see

$$\mathrm{Tot}^{\oplus}(\mathrm{Hom}(P,I))^n = \bigoplus_{p+q=n} \mathrm{Hom}\big(P_p, I^q\big)$$

This total complex is a *cochain* complex[8] because the differentials point from $\mathrm{Tot}^{\oplus}(\mathrm{Hom}(P,I))^n$ to $\mathrm{Tot}^{\oplus}(\mathrm{Hom}(P,I))^{n+1}$.

Remark 8.5.3. Let $I^\bullet$ be a cochain complex of abelian groups and let $P_\bullet$ (resp. $Q_\bullet$) be a chain complex of right (resp. left) R-modules, then there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathrm{Tot}^{\oplus}(P \otimes Q), I) \cong \mathrm{Hom}_R\big(P, \mathrm{Tot}^{\Pi}(\mathrm{Hom}_{\mathbf{Ab}}(Q, I))\big).$$

Theorem 8.5.4 (Balancing of Ext). For all n,

$$\mathrm{Ext}^n_R(A,B) = R^n \mathrm{Hom}_R(A,-)(B) \cong R^n \mathrm{Hom}_R(-,B)(A)$$

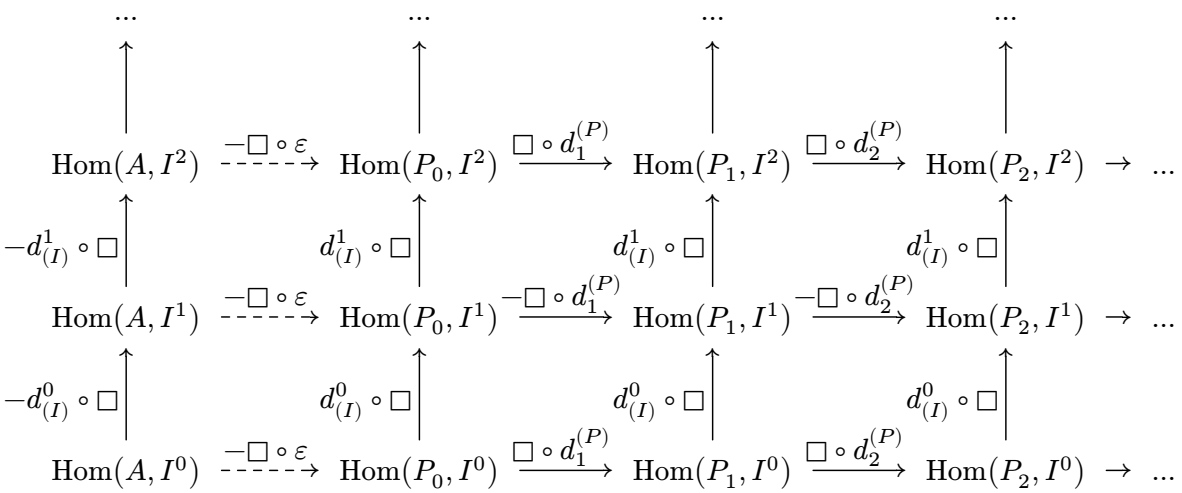
Proof. (Weibel 1994, Theorem 2.7.6, p.63). Take projective resolution $P_\bullet \xrightarrow{\varepsilon} A$ and injective resolution $B \xrightarrow{\eta} I^\bullet$. We can view A and B as complexes concentrated at degree 0. We can form double cochain complexes $\mathrm{Hom}(P,I)$, $\mathrm{Hom}(A,I)$ and $\mathrm{Hom}(P,B)$. As in the proof of Theorem 8.4.4, we need to show the maps on Hom cochain complexes

$$f : \mathrm{Hom}(A,I) \to \mathrm{Tot}^{\oplus}(\mathrm{Hom}(P,I))$$
$$g : \mathrm{Hom}(P,B) \to \mathrm{Tot}^{\oplus}(\mathrm{Hom}(P,I))$$

are quasi-isomorphisms. This is equivalent to $\mathrm{cone}(f)$ and $\mathrm{cone}(g)$ being acyclic by (the dual of) Corollary 8.2.3.

Let C be the double complex $\mathrm{Hom}(P,I)$ with $\mathrm{Hom}(A,I)$ added to the column $p=-1$ using $\varepsilon : P_0 \to A$. We make it so that every added differential has a minus sign, as shown in the diagram.

[8] In fact, whether a total complex is a chain complex or a cochain complex can seem arbitrary, because this actually depends on how we index the diagonals. Here we see the Hom total complex as a cochain complex because it is more convenient in later proofs.



We observe that $\mathrm{cone}(f) \cong \mathrm{Tot}^{\oplus}(C)$ (both their terms and differentials match). Every $\mathrm{Hom}(-, I^q)$ is exact, so every row of C is exact, then we can see that $\mathrm{Tot}^{\oplus}(C)$ is acyclic by Lemma 8.3.10. Similarly, we can show that $\mathrm{cone}(g)$ is acyclic. Then applying cohomology yields

$$\begin{aligned} R^* \operatorname{Hom}(A, -)(B) &= H^* \operatorname{Hom}(A, I) \\ &\cong H^* \mathrm{Tot}^{\oplus}(\operatorname{Hom}(P, I)) \\ &\cong H^* \operatorname{Hom}(P, B) = R^* \operatorname{Hom}(-, B)(A). \end{aligned}$$

■

Now that we have gained some experience with non-canonically ordered double complexes, we introduce another form of a Hom double complex.

Definition 8.5.5. Given two chain complexes $\left(P_\bullet, d^{(P)}\right)$ and $\left(Q_\bullet, d^{(Q)}\right)$, we can form the **Hom double complex**

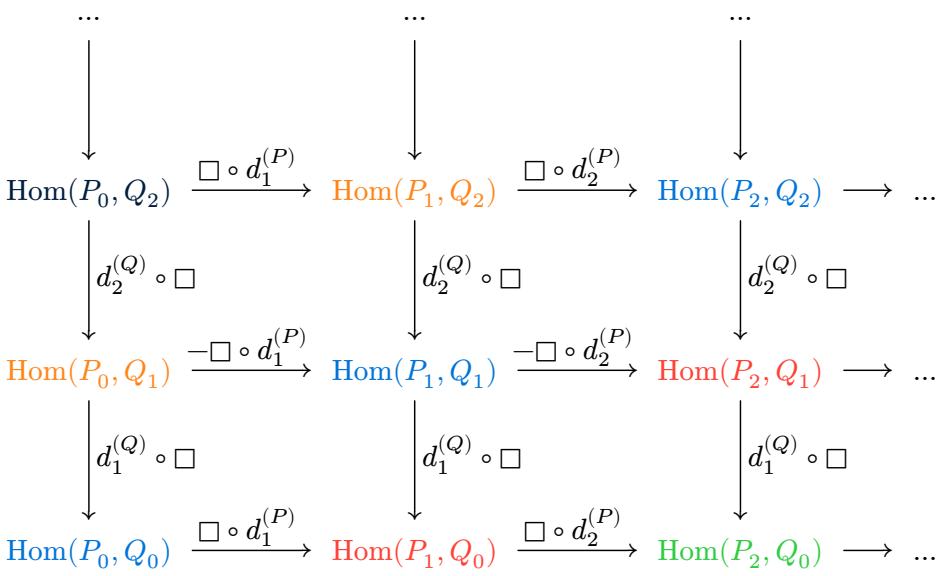
$$\mathrm{Hom}(P_\bullet, Q_\bullet) = \left\{\mathrm{Hom}\left(P_p, Q_q\right)\right\}_{p,q}$$

with differentials

$$\begin{aligned} d_{p,q}^h(f) &= (-1)^q f \circ d_{p+1}^{(P)} \in \mathrm{Hom}\left(P_{p+1}, Q_q\right) \\ d_{p,q}^v(f) &= d_q^{(Q)} \circ f \in \mathrm{Hom}\left(P_p, Q_{q-1}\right) \end{aligned}$$

for $f \in \mathrm{Hom}\left(P_p, Q_q\right)$. Then we define the **Hom cochain complex** as $\mathrm{Tot}^{\Pi}(\mathrm{Hom}(P, Q))$.

We draw the (non-canonically ordered) double complex $\mathrm{Hom}(P, Q)$ as follows. Note that each row is a cochain complex, while each column is a chain complex.



The n-th term of the total cochain complex is

$$[\mathrm{Tot}^{\Pi}(\mathrm{Hom}(P_\bullet, Q_\bullet))]^n = \prod_{p \geq \max\{0,n\}} \mathrm{Hom}\big(P_p, Q_{p-n}\big),$$

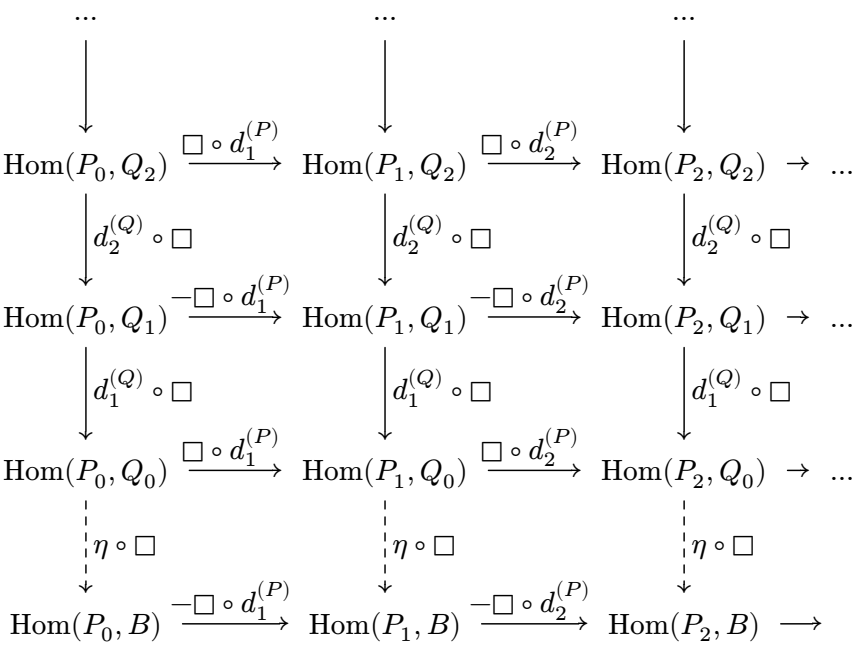
which is the product of infinitely many terms.

It turns out that this construction leads to a further way to compute Ext:

Theorem 8.5.6. Let $P_\bullet \to A$ and $Q_\bullet \to B$ be projective resolutions, then

$$\mathrm{Ext}^n_R(A, B) \cong H^n \mathrm{Tot}^{\Pi}(\mathrm{Hom}_R(P, Q)).$$

Proof. (Monnet and Kremnitzer 2021, Lemma 8.16). The proof is similar to previous ones, so we present it briefly here.



Let C be the double complex obtained by adding $\mathrm{Hom}(P, B)$ to the row $q = -1$ of the double complex $\mathrm{Hom}(P, Q)$. Since each P_p is projective, $\mathrm{Hom}(P_p, -)$ is exact and so each column of C is exact. C can be turned into a (canonically ordered) upper half-plane complex (by reflecting it to the "second quadrant"), so Lemma 8.3.8 applies and $\mathrm{Tot}^{\Pi}(C)$ is acyclic. Again, observe that $\mathrm{Tot}^{\Pi}(C) \cong \mathrm{cone}(f)$ where

$$f : \mathrm{Tot}^{\Pi}(\mathrm{Hom}(P, Q)) \to \mathrm{Hom}(P, B)$$

is the cochain map induced by $\eta : Q_\bullet \to B$. Hence f is a quasi-isomorphism, but $H^* \mathrm{Hom}(P, B) \cong \mathrm{Ext}^*_R(A, B)$ by the proof of Theorem 8.5.4, so the result follows. ■

9. Ring Structure on Ext

9.1. Reinterpreting Ext

Let $A, B \in R\text{-}\mathbf{Mod}$ with projective resolutions $P_\bullet \to A$ and $Q_\bullet \to B$. Write the total cochain complex $\mathrm{Tot}^\Pi(\mathrm{Hom}(P, Q)) =: T$. Then Theorem 8.5.6 implies that

$$\mathrm{Ext}^n_R(A, B) \cong H^n(T).$$

Recall that $T^n = \prod \mathrm{Hom}_R(P_i, Q_{i-n}) \in \mathbf{Ab}$. If $\varphi \in T^n$, then we can write $\varphi = \prod \varphi_i$ where $\varphi_i : P_i \to Q_{i-n}$. Therefore, an element φ in T^n can be seen as a chain map $P_\bullet \to Q_\bullet[-n]$ once we can show that φ commutes with the differentials of $P_\bullet$ and $Q_\bullet[-n]$. We will show that this commuting condition is (almost) equivalent to φ being a cocycle in T^n. Further, if φ is a chain map $P_\bullet \to Q_\bullet[-n]$, then φ being null homotopic is (almost) equivalent to φ being a coboundary in T^n. These equivalences ultimately give rise to a new interpretation of Ext.

The word "almost" in the last paragraph is due to some sign issues of φ. To address this, we define

$$\varepsilon_i = \begin{cases} 1 & \text{if } i \equiv 0, 3 \pmod 4 \\ -1 & \text{if } i \equiv 1, 2 \pmod 4 \end{cases}.$$

Then we define $\tilde{\varphi}_i = \varepsilon_{n-i}\varphi_i$ for each $\varphi_i : P_i \to Q_{i-n}$, and $\tilde{\varphi} = \prod \tilde{\varphi}_i \in T^n$. The following is some simple observations of the definitions.

Lemma 9.1.1. We have the following:
- $\varepsilon_i^2 = 1$;
- $\varepsilon_i \varepsilon_{i-1} = (-1)^i$;
- $\varphi_i = \varepsilon_{n-i}\tilde{\varphi}_i$;
- The map $\varphi \mapsto \tilde{\varphi}$ is an automorphism of T^n.

It turns out that we need to replace some φ with $\tilde{\varphi}$ in our previous claims, as in the following.

Proposition 9.1.2. $\varphi \in T^n$ is a cocycle in $Z^n(T)$ if and only if $\tilde{\varphi}$ can be seen as a chain map $P_\bullet \to Q_\bullet[-n]$.

Proof. As discussed above, the latter is equivalent to $\tilde{\varphi}$ commuting with the differentials, i.e., for all i,

$$\tilde{\varphi}_i \circ d^{(P)}_{i+1} = d^{(Q)}_{i+1-n} \circ \tilde{\varphi}_{i+1}. \tag{1}$$

Using the definition of $\tilde{\varphi}$, Equation 1 can reduced to

$$\varphi_i \circ d^{(P)}_{i+1} = (-1)^{i+1-n} d^{(Q)}_{i+1-n} \circ \varphi_{i+1}. \tag{2}$$

On the other hand, $\varphi \in T^n$ is a cocycle, i.e., $d^{(T)}_n(\varphi) = 0$, if and only if for all i,

$$d^h(\varphi_i) + d^v(\varphi_{i+1}) = 0. \tag{3}$$

Using the definition of d^h and d^v, we see that Equation 3 is equivalent to Equation 2. ∎

Proposition 9.1.3. $\varphi \in Z^n(T)$ is a coboundary in $B^n(T)$ if and only if the chain map $\tilde{\varphi} : P_\bullet \to Q_\bullet[-n]$ is null homotopic.

Proof. "⇒". Suppose φ is a coboundary. Then $\varphi = d(\psi)$ for some $\psi = \prod \psi_i \in T^{n-1}$, where $\psi_i : P_i \to Q_{i-n+1}$. We also write $\tilde{\psi}_i = \varepsilon_{i-n+1}\psi_i$.

$$\begin{array}{ccc} & & \mathrm{Hom}(P_i, Q_{i-n+1}) \\ & & \Big\downarrow{\scriptstyle d^{(Q)}_{i-n+1} \circ \square} \\ \mathrm{Hom}(P_{i-1}, Q_{i-n}) & \xrightarrow{(-1)^{i-n}\square \circ d^{(P)}_i} & \mathrm{Hom}(P_i, Q_{i-n}) \end{array}$$

Considering each φ_i, we see that

$$\varphi_i = d^h(\psi_{i-1}) + d^v(\psi_i) = (-1)^{i-n}\psi_{i-1} \circ d^{(P)}_i + d^{(Q)}_{i-n+1} \circ \psi_i.$$

Using Lemma 9.1.1, we can reduce this to

$$\tilde{\varphi}_i = (-1)^{i-n}\tilde{\psi}_{i-1} \circ d^{(P)}_i + (-1)^{i-n+1} d^{(Q)}_{i-n+1} \circ \tilde{\psi}_i.$$

Then we see that $(-1)^{i-n+1}\tilde{\psi}_i : P_i \to Q_{i-n+1}$ is a chain homotopy between $\tilde{\varphi}$ and 0.

"⇐". Suppose that $h_i : P_i \to Q_{i-n+1}$ is a chain homotopy between $\tilde{\varphi}$ and 0 for each i, so that

$$\tilde{\varphi}_i = h_{i-1} \circ d^{(P)}_i + d^{(Q)}_{i-n+1} \circ h_i.$$

Then we can define for each i,

$$\tilde{\psi}_i = (-1)^{i-n+1} h_i, \quad \psi_i = \varepsilon_{i-n+1}\tilde{\psi}_i,$$

which gives $\psi = \prod \psi_i \in T^{n-1}$. Again with Lemma 9.1.1, we can reveal that $d(\psi) = \varphi$, so φ is a coboundary. ∎

Corollary 9.1.4. $\mathrm{Ext}^n_R(A, B)$ is isomorphic to the chain homotopy classes of chain maps $P_\bullet \to Q_\bullet[-n]$.

Proof. $\mathrm{Ext}^n_R(A, B) \cong H^n(T) = Z^n(T)/B^n(T)$, but now $Z^n(T)$ is isomorphic to the group of chain maps $P_\bullet \to Q_\bullet[-n]$ and $B^n(T)$ is isomorphic to the subgroup of null homotopic chain maps. ∎

9.2. Yoneda Product

Proposition 9.2.1. Given left R-modules A, B, C, for any i, j, there is a well-defined map, called the **Yoneda product**,

$$\smile : \mathrm{Ext}^i_R(A, B) \times \mathrm{Ext}^j_R(B, C) \to \mathrm{Ext}^{i+j}_R(A, C),$$

which is associative and biadditive.

Proof. Write projective resolutions $P_\bullet \to A$, $Q_\bullet \to B$, $T_\bullet \to C$. Take $x \in \mathrm{Ext}^i_R(A,B)$ and $y \in \mathrm{Ext}^j_R(B,C)$. By Corollary 9.1.4, we see that x (resp. y) corresponds to some $[\varphi]$ (resp. $[\psi]$) which is a chain homotopy class (of chain maps) $P_\bullet \to Q_\bullet[-i]$ (resp. $Q_\bullet \to T_\bullet[-j]$). Note that

$$\mathrm{Hom}(Q_\bullet, T_\bullet[-j]) \cong \mathrm{Hom}(Q_\bullet[-i], T_\bullet[-i-j])$$

because the translation functor is an isomorphism on $\mathrm{Ch}(R\text{-}\mathbf{Mod})$ and preserves chain homotopy. Hence $[\psi]$ can be also viewed as a chain homotopy class $Q_\bullet[-i] \to T_\bullet[-i-j]$. Since chain homotopy commutes with composition, we can obtain $[\psi \circ \varphi] = [\psi] \circ [\varphi]$, which is a chain homotopy class $P_\bullet \to T_\bullet[-i-j]$. Then we define $x \smile y \in \mathrm{Ext}^{i+j}_R(A,C)$ to be the corresponding element of $[\psi \circ \varphi]$. We can see that $\smile$ is associative and biadditive because the composition of chain maps is associative and biadditive. ■

Corollary 9.2.2. For any $A, B \in R\text{-}\mathbf{Mod}$,

$$\mathrm{Ext}^*_R(A,A) = \bigoplus_i \mathrm{Ext}^i_R(A,A)$$

is a graded ring, and

$$\mathrm{Ext}^*_R(A,B) = \bigoplus_i \mathrm{Ext}^i(A,B)$$

is a graded module over $\mathrm{Ext}^*_R(A,A)$.

Example 9.2.3. Let k be a field and $R = k[x]/(x^2)$. View k as the R-module R/xR. We now calculate the graded ring structure of $\mathrm{Ext}^*_R(k,k)$.

First, observe that we have a projective resolution of k:

$$\ldots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \to k \to 0,$$

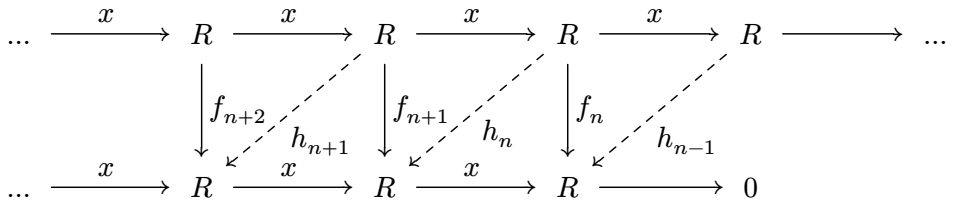
which we denote as $P_\bullet \to k \to 0$. If we apply $\mathrm{Hom}_R(-,k)$ to $P_\bullet$, then $\mathrm{Hom}_R(R,k) \cong k$ and all differentials vanish, so we can quickly use the balancing of Ext to reveal that $\mathrm{Ext}^n_R(k,k) \cong k$ for all $n \geq 0$. We however need to find the generator explicitly for the ring structure.

Now by Corollary 9.1.4, $\mathrm{Ext}^n_R(k,k)$ is isomorphic to the chain homotopy classes of chain maps $P_\bullet \to P_\bullet[-n]$. A chain map $P_\bullet \to P_\bullet[-n]$ is a collection of R-homomorphisms $f_i : R \to R$ for $i \geq n$ such that the following diagram commutes:

$$\begin{array}{ccccccccccc}
\ldots & \xrightarrow{x} & R & \xrightarrow{x} & R & \xrightarrow{x} & R & \xrightarrow{x} & R & \longrightarrow & \ldots \\
 & & \downarrow f_{n+2} & & \downarrow f_{n+1} & & \downarrow f_n & & & & \\
\ldots & \xrightarrow{x} & R & \xrightarrow{x} & R & \xrightarrow{x} & R & \longrightarrow & 0 & &
\end{array}$$

Note that each f_i is uniquely determined by $f_i(1) \in R$. We write $f_i(1) = a_i + b_i x$ for $a_i, b_i \in k$ and then we only need to look at a_i, b_i for each i. The commutativity of the above diagram indicates that $x(a_i + b_i x) = x(a_{i+1} + b_{i+1} x)$ for each i, i.e. $a_i = a_{i+1}$, so all the a_i's are equal.

We now consider when the chain map f_* is null homotopic. By definition, that is when there is a collection of R-homomorphisms $h_i : R \to R$ for $i \geq n-1$ such that $f_i = h_{i-1} \circ x + x \circ h_i$.



In particular, this indicates that $a_i + b_i x = f_i(1) = h_{i-1}(x) + x h_i(1) = x(h_{i-1}(1) + h_i(1))$, and thus $a_i = 0$ for all i.

We claim that $a_i = 0$ for all i is a sufficient condition for f_* to be null homotopic. In this case, $f_i(1) = b_i x$ for each $i \geq n$. We construct $h_i : R \to R$ for $i \geq n-1$ as follows: $h_{n-1} = 0$ and $h_i(1) = b_i - h_{i-1}(1)$ for all $i \geq n$. Then f_* is null homotopic via h.

Therefore f_* is null homotopic if and only if $f_i(1) \equiv 0 \pmod{x}$ for all i. So "chain maps modulo homotopy" is the same as "chain maps modulo x". In other words, two chain maps $f_*, g_* : P_\bullet \to P_\bullet[-n]$ are chain homotopic if and only if $f_i(1) \equiv g_i(1) \pmod{x}$ for all i. But we have established that $f_i(1)$'s are all equal modulo x. So $\mathrm{Ext}^n_R(k,k) \cong k \cdot f_*^{(n)}$, where $f_*^{(n)}$ is the chain map $P_\bullet \to P_\bullet[-n]$ with $f_i^{(n)}(1) = 1$ for all $i \geq n$.

It is clear that $f_*^{(m)} \circ f_*^{(n)} = f_*^{(m+n)}$, so the Yoneda product gives $f_*^{(m)} \smile f_*^{(n)} = f_*^{(m+n)}$. Set $y = f_*^{(1)}$, then for each n, $f_*^{(n)} = y^n$ (using Yoneda product) and thus $\mathrm{Ext}^n_R(k,k) \cong k \cdot y^n$. So as a graded ring, $\mathrm{Ext}^*_R(k,k) \cong k[y]$ where y has degree 1.

10. Tor and Flatness

10.1. Flat Modules

Definition 10.1.1. A left R-module B is **flat** if $(- \otimes_R B)$ is exact. A right R-module A is **flat** if $(A \otimes_R -)$ is exact.

Proposition 10.1.2. Let B a left R-module. The followings are equivalent:
(1) B is flat;
(2) $\mathrm{Tor}_n^R(A, B) = 0$ for all $n > 0$ and all A;
(3) $\mathrm{Tor}_1^R(A, B) = 0$ for all A.

Similarly, let A be a right R-module. The followings are equivalent:
(1) A is flat;
(2) $\mathrm{Tor}_n^R(A, B) = 0$ for all $n > 0$ and all B;
(3) $\mathrm{Tor}_1^R(A, B) = 0$ for all B.

Proof. (Monnet and Kremnitzer 2021, Lemma 6.26). We prove the left R-module case.

(1) $\Rightarrow$ (2). Suppose that B is flat. Let $F_\bullet \to A$ be a free resolution of A. Since $(- \otimes_R B)$ is exact, the sequence

$$\ldots \to F_2 \otimes_R B \to F_1 \otimes_R B \to F_0 \otimes_R B \to A \otimes_R B \to 0$$

is exact, so the homology of

$$\ldots \to F_2 \otimes_R B \to F_1 \otimes_R B \to F_0 \otimes_R B \to 0$$

vanishes in positive degree.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). For any short exact sequence $0 \to X \to Y \to A \to 0$ in $\mathbf{Mod}$-R, we have the long exact sequence of Tor by Proposition 8.1.7,

$$0 = \mathrm{Tor}_1^R(A, B) \to X \otimes_R B \to Y \otimes_R B \to A \otimes_R B \to 0,$$

which shows that $(- \otimes_R B)$ is exact.

Note that right R-module case relies on the balancing of Tor, but the proof is very similar. ∎

Corollary 10.1.3. Every projective module is flat. In particular, every free module is flat.[9]

Proof. If a left R-module P is projective, by Theorem 8.4.4 and Corollary 7.2.5, $\mathrm{Tor}_n^R(A, P) \cong L_n(A \otimes_R -)(P) = 0$ for all A and all $n \geq 1$. Then applying Proposition 10.1.2 gives the result. The case where P is a projective right R-module is similar. ∎

[9] We have already proven this claim in Lemma 8.4.3, because we needed it for the balancing of Tor (Theorem 8.4.4). This second proof actually relies on the balancing of Tor so we could not use it previously, but it is presented here regardless.

Definition 10.1.4. A category I is called **filtered** if
(1) I is non-empty;
(2) For any $i, j \in I$, there exists $k \in I$ with morphisms $f : i \to k$ and $g : j \to k$;
(3) For any $i, j \in I$ with a pair of morphisms $u, v : i \to j$, there exists $k \in I$ with morphism $w : j \to k$ such that $w \circ u = w \circ v$.

Example 10.1.5. A non-empty partially ordered set (poset) I, viewed as a small category, is **filtered** if for any $i, j \in I$, there exists k such that $i \leq k$ and $j \leq k$. This is because condition (3) above is automatically satisifed, as there is at most one morphism $i \to j$ for any $i, j \in I$.

Proposition 10.1.6. Let I be a filtered category. Then the functor

$$\operatorname{colim}_I : \operatorname{Fun}(I, R\text{-}\mathbf{Mod}) \to R\text{-}\mathbf{Mod}$$

is exact.

Proof. (Weibel 1994, Theorem 2.6.15, p.57). ∎

Remark 10.1.7. colim_I is not a exact functor in general if I is not filtered.

Notation 10.1.8. Let I be a small category and $A : I \to R\text{-}\mathbf{Mod}$ be a diagram. We denote $A_i = A(i)$ for each $i \in I$ and we would write $\operatorname{colim}_I A_i$ to mean $\operatorname{colim}_I A$.

Corollary 10.1.9. Let I be a filtered category and $A : I \to \mathbf{Mod}\text{-}R$ be a diagram. Let $B \in R\text{-}\mathbf{Mod}$. Then $\operatorname{Tor}_n^R(\operatorname{colim}_I A_i, B) \cong \operatorname{colim}_I \operatorname{Tor}_n^R(A_i, B)$. In other words, filtered colimits commute with Tor.

Proof. Let $P_\bullet \to B$ be a projective resolution. Then

$$\begin{aligned}
\operatorname{Tor}_n^R(\operatorname{colim}_I A_i, B) &= H_n((\operatorname{colim}_I A_i) \otimes_R P) \\
&\cong H_n(\operatorname{colim}_I (A_i \otimes_R P)) \\
&\cong \operatorname{colim}_I H_n(A_i \otimes_R P) \\
&= \operatorname{colim}_I \operatorname{Tor}_n^R(A_i, B),
\end{aligned}$$

where at each step we respectively use the definition of Tor, that colimits commute with tensor products, that colim_I is exact and thus commutes with homology, and the definition of Tor again. ∎

Corollary 10.1.10. Let I be a filtered category and $A : I \to \mathbf{Mod}\text{-}R$ be a diagram. Suppose A_i is flat for all $i \in I$. Then $\operatorname{colim}_I A_i$ is also flat. In other words, a filtered colimit of flat R-modules is flat.

Proof. Take any $B \in R\text{-}\mathbf{Mod}$. Since each A_i is flat, we know that $\operatorname{Tor}_1(A_i, B) = 0$ by Proposition 10.1.2. Then

$$\operatorname{Tor}_1(\operatorname{colim}_I A_i, B) = \operatorname{colim}_I \operatorname{Tor}_1(A_i, B) = 0$$

by Corollary 10.1.9, so $\operatorname{colim}_I A_i$ is also flat by Proposition 10.1.2 again. ■

Example 10.1.11. Let $s \in R$ be a central element of ring R, then the localisation $R[s^{-1}]$ is a flat R-module. To generalise, for a central multiplicatively closed set $S \subset Z(R)$, we can form $R[S^{-1}]$, which is a flat R-module as well.

Proof. (Weibel 1994, Theorem 3.2.2, p.69). ■

We now take a look at the case in $\mathbf{Ab}$ and we shall show that a module in $\mathbf{Ab}$ is flat if and only if it is torsion-free.

Lemma 10.1.12. Let $B \in \mathbf{Ab}$ and $p \in \mathbb{Z}$. Then $\operatorname{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) = B/pB$ and $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) = \{b \in B : pb = 0\}$.

Proof. Use the definition of Tor, the projective resolution $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$, and $B \otimes_{\mathbb{Z}} \mathbb{Z} \cong B$. ■

Lemma 10.1.13. An abelian group is a filtered colimit of its finitely generated subgroups.

Proof. $A = \cup A_i = \operatorname{colim}_I A_i$ where I is a filtered poset representing the inclusion relations of the finitely generated subgroups of A. ■

Lemma 10.1.14. Let $A, B \in \mathbf{Ab}$. Then $\operatorname{Tor}_1^{\mathbb{Z}}(A, B)$ is a torsion abelian group.

Remark 10.1.15. This is likely why Tor is called Tor.

Proof. By writing $A = \operatorname{colim}_I A_i$ for finitely genreated subgroups A_i of A, we see that it suffices to show that each $\operatorname{Tor}_1^{\mathbb{Z}}(A_i, B)$ is torsion. We can write A_i as a direct sum of its torsion part and free part using the classification theorem for finitely generated abelian groups, i.e. $A_i = \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p_m\mathbb{Z} \oplus \mathbb{Z}^r$. Notice that Tor commutes with direct sums and the free part $\mathbb{Z}^r$ vanishes with Tor, so

$$\operatorname{Tor}_1^{\mathbb{Z}}(A_i, B) \cong \bigoplus_{k=1}^{m} \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p_k\mathbb{Z}, B) \cong \bigoplus_{k=1}^{m} B/p_kB,$$

which is clearly a torsion abelian group. ■

Lemma 10.1.16. Let $B \in \mathbf{Ab}$. Then $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B)$ is the torsion subgroup of B, i.e. $\{b \in B :$ there exists $n \in \mathbb{Z}$ such that $nb = 0\}$.

Proof. $\mathbb{Q}/\mathbb{Z}$ can be written as the filtered colimit $\mathbb{Q}/\mathbb{Z} \cong \mathrm{colim}_I \mathbb{Z}/p\mathbb{Z}$, where I is the poset representing the divisibility of natural numbers. Then

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B) \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{colim}_I \mathbb{Z}/p\mathbb{Z}, B) \cong \mathrm{colim}_I \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) \cong \mathrm{colim}_I \{b \in B : pb = 0\},$$

which is the torsion subgroup of B. ∎

Proposition 10.1.17. A $\mathbb{Z}$-module is flat if and only if it is torsion-free.

Proof. Let A be a torsion-free abelian group, then $A = \cup A_i$ where A_i are finitely generated subgroups of A. Then each A_i is free and hence flat. By Corollary 10.1.10, A is also flat.

On the other hand, if A is flat, then $\mathrm{Tor}_1^{\mathbb{Z}}(-, A) = 0$; in particular, $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A) = 0$, so the torsion subgroup of A is trivial. ∎

10.2. Flat Resolutions

Definition 10.2.1. A resolution $F_\bullet \to A$ is a **flat resolution** of A if F_n is a flat module for each n.

Lemma 10.2.2 (Flat Resolution Lemma). Let $F_\bullet \to A$ be a flat resolution of A. Then

$$\mathrm{Tor}_*^R(A, B) \cong H_*(F_\bullet \otimes_R B)$$

Similarly, if $F'_\bullet \to B$ is a flat resolution, then

$$\mathrm{Tor}_*^R(A, B) \cong H_*(A \otimes_R F'_\bullet)$$

Proof. By induction. When $n = 0$, we need to show that $A \otimes_R B \cong H_0(F_\bullet \otimes_R B)$. We see that

$$H_0(F_\bullet \otimes_R B) = \frac{F_0 \otimes_R B}{\mathrm{Im}(F_1 \otimes_R B)} = \mathrm{Coker}(F_1 \otimes_R B \to F_0 \otimes_R B)$$

On the other hand, $A = \mathrm{Coker}(F_1 \to F_0)$, and since $(- \otimes_R B)$ is right exact and preserves cokernels, we get the result.

When $n \geq 1$, we have the short exact sequence

$$0 \to K \overset{\ker(\varepsilon)}{\to} F_0 \overset{\varepsilon}{\to} A \to 0$$

If we write $E_\bullet = (\ldots \to F_2 \to F_1 \to 0)$, then $E_\bullet \overset{d_1}{\to} K$ is a flat resolution of K, where $d_1 : F_1 \to F_0$ has $\mathrm{Im}(d_1) = \mathrm{Ker}(\varepsilon) = K$ by the exactness at F_0 of the resolution $F_\bullet \to A$. Now we can write the long exact sequence induced by Tor:

$$\dots \to \operatorname{Tor}_n(F_0, B) \to \operatorname{Tor}_n(A, B) \to \operatorname{Tor}_{n-1}(K, B) \to \operatorname{Tor}_{n-1}(F_0, B) \to \dots$$

But $\operatorname{Tor}_n(F_0, B) = 0$ for $n \geq 1$ by Proposition 10.1.2. Thus,

$$\operatorname{Tor}_n(A, B) \cong \begin{cases} \operatorname{Ker}(K \otimes B \to F_0 \otimes B) & n = 1 \\ \operatorname{Tor}_{n-1}(K, B) & n \geq 2 \end{cases}$$

For $n = 1$, notice that $K \otimes_R B \cong H_0(E_\bullet \otimes_R B) = (F_1 \otimes_R B)/\operatorname{Im}(F_2 \otimes_R B)$ by applying the case $n = 0$ to $E_\bullet \to K$, and hence we have

$$\operatorname{Tor}_1(A, B) \cong \operatorname{Ker}(K \otimes B \to F_0 \otimes B) = \operatorname{Ker}\left(\frac{F_1 \otimes B}{\operatorname{Im}(F_2 \otimes B)} \to F_0 \otimes B\right) = H_1(F_\bullet \otimes B)$$

For $n \geq 2$,

$$\operatorname{Tor}_n(A, B) \cong \operatorname{Tor}_{n-1}(K, B) \cong H_{n-1}(E_\bullet \otimes B) = H_n(F_\bullet \otimes B)$$

by applying the induction hypothesis to $E_\bullet \to K$. ∎

Remark 10.2.3. Why have we not defined Tor with flat resolutions in the first place? The problem is that we have to show it is well defined regardless of the choice of flat resolutions. This may not be as convenient as using projective resolutions. Nevertheless, now we are free to use flat resolutions, a larger class than projective resolutions, for calculations.

A generalisation to flat modules is the following.

Definition 10.2.4. If F is a right exact functor, an object Q is F-**acyclic** if $L_i F(Q) = 0$ for all $i \neq 0$.

Proposition 10.2.5. If $Q_\bullet \to A$ is a resolution where Q_n is F-acyclic for all n, then $L_i F(A) = H_i(F(Q_\bullet))$.

Proof. The proof is exactly the same as above. ∎

10.3. Universal Coefficient Theorem

Let $P_\bullet$ be a chain complex of right R-modules and let M be a left R-module. In this section, we investigate how the homology of $P_\bullet$ is related to the homology of $P_\bullet \otimes M$, under certain flatness assumptions about $P_\bullet$. We first need an auxiliary result.

Lemma 10.3.1. Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $\mathbf{Mod}$-R such that Y and Z are flat, then X is also flat.

Proof. Let $B \in R$-$\mathbf{Mod}$. Write the long exact sequence induced by Tor:

$$\dots \to \operatorname{Tor}^R_{n+1}(Z, B) \to \operatorname{Tor}^R_n(X, B) \to \operatorname{Tor}^R_n(Y, B) \to \dots$$

Since Y and Z are flat, $\operatorname{Tor}^R_n(Y, B) = \operatorname{Tor}^R_n(Z, B) = 0$ when $n \geq 1$ by Proposition 10.1.2. So $\operatorname{Tor}^R_n(X, B) = 0$ for any $n \geq 1$ and X is also flat. ∎

The main result is the following.

Theorem 10.3.2 (Künneth Formula). Let $P_\bullet$ be a chain complex of flat right R-modules such that each submodule $d(P_n)$ of P_{n-1} is also flat. Then for every n and every left R-module M, there is a short exact sequence

$$0 \to H_n(P) \otimes_R M \to H_n(P_\bullet \otimes_R M) \to \operatorname{Tor}_1^R(H_{n-1}(P), M) \to 0.$$

Proof. (Rotman 2009, Theorem 7.55), (Weibel 1994, Theorem 3.6.1). Let $Z_n = \operatorname{Ker}\left(P_n \xrightarrow{d} P_{n-1}\right)$, then we have a short exact sequence

$$0 \to Z_n \to P_n \to d(P_n) \to 0. \tag{4}$$

Since P_n and $d(P_n)$ are both flat, Lemma 10.3.1 shows that Z_n is also flat. Also, since $d(P_n)$ is flat, $\operatorname{Tor}_1^R(d(P_n), M) = 0$ by Proposition 10.1.2, so a long exact sequence induced by Short Exact Sequence 4 gives

$$0 \to Z_n \otimes M \to P_n \otimes M \to d(P_n) \otimes M \to 0,$$

from which we yield a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes M \to P_\bullet \otimes M \to d(P)_\bullet \otimes M \to 0.$$

We now look at the long exact sequence induced by homology:

$$\begin{aligned} &\dots \to H_{n+1}\left(d(P)_\bullet \otimes M\right) \xrightarrow{\partial} H_n(Z_\bullet \otimes M) \\ &\to H_n(P_\bullet \otimes M) \to H_n\left(d(P)_\bullet \otimes M\right) \xrightarrow{\partial} H_{n-1}(Z_\bullet \otimes M) \to \dots \end{aligned} \tag{5}$$

Note that the differentials on the chain complexes $Z_\bullet$ and $d(P)_\bullet$ are all zero, and hence the differentials on $Z_\bullet \otimes M$ and $d(P)_\bullet \otimes M$ are also all zero, which gives

$$H_n\left(d(P)_\bullet \otimes M\right) = d(P_n) \otimes M \quad \text{and} \quad H_n(Z_\bullet \otimes M) = Z_n \otimes M.$$

Hence Long Exact Sequence 5 now becomes

$$\dots \to d(P_{n+1}) \otimes M \xrightarrow{\partial_{n+1}} Z_n \otimes M \to H_n(P_\bullet \otimes M) \to d(P_n) \otimes M \xrightarrow{\partial_n} Z_{n-1} \otimes M \to \dots$$

Apply Theorem 6.3.2 and we can find the connecting homomorphism $\partial_{n+1} = i_n \otimes \mathrm{id}_M$, where $i_n : d(P_{n+1}) \to Z_n$ is the inclusion map. By Proposition 3.5.8, we have a short exact sequence

$$0 \to \operatorname{Coker}(i_n \otimes \mathrm{id}_M) \to H_n(P_\bullet \otimes M) \to \operatorname{Ker}(i_{n-1} \otimes \mathrm{id}_M) \to 0. \tag{6}$$

Now it remains to calculate the two flanking terms of Short Exact Sequence 6. Note that we have a flat resolution for $H_n(P_\bullet)$:

$$0 \to d(P_{n+1}) \xrightarrow{i_n} Z_n \to H_n(P_\bullet) \to 0.$$

By Lemma 10.2.2, $\operatorname{Tor}_*^R(H_n(P_\bullet), M)$ is the homology of the following chain complex:

$$D_\bullet = \left(\dots \to 0 \to d(P_{n+1}) \otimes M \xrightarrow{i_n \otimes \mathrm{id}_M} Z_n \otimes M \to 0\right)$$

Hence

$$H_n(P_\bullet) \otimes M = \mathrm{Tor}_0(H_n(P_\bullet), M) = H_0(D) = \mathrm{Coker}(i_n \otimes \mathrm{id}_M)$$

and

$$\mathrm{Tor}_1^R(H_n(P_\bullet), M) = H_1(D) = \mathrm{Ker}(i_n \otimes \mathrm{id}_M).$$

Combining the last two equations with Short Exact Sequence 6 gives the result. ■

Remark 10.3.3. These are the games you play with the machine.

In particular, we can consider the case of $\mathbf{Ab}$, where we have the following:

Theorem 10.3.4 (Universal Coefficient Theorem). Let $P_\bullet$ be a chain complex of free abelian groups, then for every n and every M, the Künneth short exact sequence splits, so

$$H_n(P_\bullet \otimes M) = (H_n(P) \otimes M) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(P), M)$$

The split is not canonical.

Proof. (Rotman 2009, Corollary 7.56, p. 450), (Weibel 1994, Theorem 3.6.2, p. 87). Since each P_n is a free abelian group, its subgroup $d(P_n)$ is also free abelian by Proposition 1.10. Therefore $d(P_n)$ is projective, so the short exact sequence

$$0 \to Z_n \overset{i_n}{\to} P_n \to d(P_n) \to 0$$

splits by Proposition 3.6.2. Applying $(- \otimes M)$ (which commutes with direct sums), we see that the short exact sequence

$$0 \to Z_n \otimes M \overset{i_n \otimes \mathrm{id}_M}{\to} P_n \otimes M \to d(P_n) \otimes M \to 0$$

also splits, so $Z_n \otimes M$ is a direct summand of $P_n \otimes M$. Now notice we have the inclusions

$$\mathrm{Im}(d_{n+1} \otimes \mathrm{id}_M) \subseteq \mathrm{Im}(i_n \otimes \mathrm{id}_M) \subseteq \mathrm{Ker}(d_n \otimes \mathrm{id}_M) \subseteq P_n \otimes M$$

By Corollary 3.5.11, $Z_n \otimes M$ is a direct summand of $\mathrm{Ker}(d_n \otimes \mathrm{id}_M)$. Modding out by $\mathrm{Im}(d_{n+1} \otimes \mathrm{id}_M)$ on both of them, again by Corollary 3.5.11, we see that

$$H_n(P) \otimes M = \frac{Z_n \otimes M}{\mathrm{Im}(d_{n+1} \otimes \mathrm{id}_M)} \tag{7}$$

is a direct summand of

$$H_n(P_\bullet \otimes M) = \frac{\mathrm{Ker}(d_n \otimes \mathrm{id}_M)}{\mathrm{Im}(d_{n+1} \otimes \mathrm{id}_M)}.$$

To see why Equation 7 holds, let $j_n : d(P_{n+1}) \to Z_n$ be the inclusion map, and by the proof of Theorem 10.3.2,

$$\begin{aligned} H_n(P) \otimes M &= \mathrm{Coker}(j_n \otimes \mathrm{id}_M : d(P_{n+1}) \otimes M \to Z_n \otimes M) \\ &= \frac{Z_n \otimes M}{\mathrm{Im}(j_n \otimes \mathrm{id}_M)} = \frac{Z_n \otimes M}{\mathrm{Im}(d_{n+1} \otimes \mathrm{id}_M)}. \end{aligned}$$

Since each P_n and $d(P_n)$ are projective and thus flat, by Theorem 10.3.2 we have a short exact sequence

$$0 \to H_n(P) \otimes M \to H_n(P_\bullet \otimes M) \to \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(P), M) \to 0,$$

which is therefore split. ∎

We also demonstrate a more general result, known as the Full Künneth Formula. Now it can be helpful to recall Definition 8.4.1, the tensor product double complex, as well as Notation 8.3.5.

Theorem 10.3.5 (Full Künneth Formula). Let $P_\bullet$ and $Q_\bullet$ be right and left R-modules, respectively. If P_n and $d(P_n)$ are flat for each n, then there is a short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(P) \otimes H_j(Q) \to H_n(P \otimes_R Q) \to \bigoplus_{i+j=n-1} \operatorname{Tor}_1^R\big(H_i(P), H_j(Q)\big) \to 0$$

Proof. (Weibel 1994, Theorem 3.6.3). Modify the proof of Theorem 10.3.2. ∎

It is worth mentioning that there is also a version of the Universal Coefficient Theorem for cohomology that involves Ext and Hom:

Theorem 10.3.6 (Universal Coefficient Theorem for Cohomology). Let $P_\bullet$ be a chain complex of projective R-modules such that each $d(P_n)$ is also projective. Then for any n and every R-module M, there is a non-canonically split exact sequence

$$0 \to \operatorname{Ext}_R^1(H_{n-1}(P), M) \to H^n(\operatorname{Hom}_R(P, M)) \to \operatorname{Hom}_R(H_n(P), M) \to 0.$$

Proof. (Weibel 1994, Theorem 3.6.5). ∎

These results yield important consequences in algebraic topology, as briefly discussed below. More can be seen in the Algebraic Topology course.

Example 10.3.7. (Weibel 1994, Application 3.6.4). Let X be a topological space. Let $C_\bullet(X)$ be the singular chain complex of X, then each $C_n(X)$ is a free abelian group. Let M be an abelian group, then we define the n-th (singular) homology of X with *coefficients* in M as

$$H_n(X; M) := H_n(C_\bullet(X) \otimes M).$$

In particular, $H_n(X) = H_n(X; \mathbb{Z})$. Then the Universal Coefficient Theorem gives

$$H_n(X; M) \cong (H_n(X) \otimes M) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), M).$$

For cohomology, we could make a similar definition, i.e.

$$H^n(X; M) := H^n(\operatorname{Hom}_{\mathbb{Z}}(C_\bullet(X), M))$$

with $H^n(X) = H^n(X; \mathbb{Z})$, and the Universal Coefficient Theorem would indicate that

$$H^n(X; M) \cong \operatorname{Hom}_{\mathbb{Z}}(H_n(X), M) \oplus \operatorname{Ext}_{\mathbb{Z}}^1(H_{n-1}(X), M).$$

(If we further assume that $M = \mathbb{Z}$ and that each $H_n(X)$ is finitely generated such that $H_n(X) \cong F_n \oplus T_n$ with free part F_n and torsion part T_n, then we can show that $H^n(X) \cong F_n \oplus T_{n-1}$.)

Let Y be another topological space with singular chain complex $C_\bullet(Y)$. By Eilenberg–Zilber theorem, $H_n(X \times Y) \cong H_n(C_\bullet(X) \otimes C_\bullet(Y))$. Then the Full Künneth Formula indicates that

$$H_n(X \times Y) \cong \left(\bigoplus_{p=1}^{n} H_p(X) \otimes H_{n-p}(Y) \right) \oplus \left(\bigoplus_{p=1}^{n} \operatorname{Tor}_1^{\mathbb{Z}}\big(H_{p-1}(X), H_{n-p}(Y)\big) \right).$$

11. Koszul Complexes and (Co)homology

We generally follow (Weibel 1994, Section 4.5). In this section, by an R-module we mean either a left or a right R-module.

11.1. Koszul Complexes

Definition 11.1.1. Let R be a ring and let $x \in Z(R)$ be a central element. Then we define the **Koszul complex** $K_\bullet(x)$ of x to be the chain complex

$$0 \to R \cdot e_x \xrightarrow{x} R \to 0$$

concentrated in degrees 1 and 0, where e_x is a symbol to denote the generator of $K_1(x)$, and the differential $R \cdot e_x \xrightarrow{x} R$ is multiplication by x, i.e. $d(e_x) = x$.

Definition 11.1.2. If $\boldsymbol{x} = (x_1, ..., x_n)$ is a finite sequence of central elements of R, then by above, we have Koszul complexes $K(x_1), ..., K(x_n)$, where we write $e_i = e_{x_i}$. Then the chain complex $K(\boldsymbol{x})$ is defined as follows. The symbols

$$e_{i_1} \wedge \cdots \wedge e_{i_p} = \underbrace{1 \otimes \cdots \otimes 1 \otimes e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes \cdots \otimes 1}_{n \text{ terms}} \quad \big(1 \le i_1 < \cdots < i_p \le n\big).$$

generate the free R-module $K_p(\boldsymbol{x})$, and the differential $K_p(\boldsymbol{x}) \to K_{p-1}(\boldsymbol{x})$ sends $e_{i_1} \wedge \cdots \wedge e_{i_p}$ to

$$\sum_{k=1}^{p} (-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_p}.$$

Remark 11.1.3. Alternatively, we could define $K(\boldsymbol{x})$ as the total tensor product complex

$$\mathrm{Tot}^{\oplus}(K(x_1) \otimes_R K(x_2) \otimes_R ... \otimes_R K(x_n))$$

by which we mean an inductive relation

$$K(\boldsymbol{x}) = \mathrm{Tot}^{\oplus}(K(x_1, x_2, ..., x_{n-1}), K(x_n)).$$

We omit the proof that this is an equivalent definition. One may also understand the alternative definition by regarding $C := K(x_1) \otimes_R ... \otimes_R K(x_n)$ as an "n-dimensional complex", a generalisation of a double complex. In particular, C has in total 2^n terms, and a typical term in C is indexed by an n-tuple of 0 and 1, whose total degree is the sum of this n-tuple. For example, if $n = 4$, C has a term

$$C_{0,1,1,0} = R \otimes_R (R \cdot e_2) \otimes_R (R \cdot e_3) \otimes_R R,$$

which has total degree 2 and is generated by $e_2 \wedge e_3$. Then $K(\boldsymbol{x})$ is the total complex of C, where $K_p(\boldsymbol{x})$ is the direct sum of all terms in C which has total degree p. For example, when $n = 4$, $K_2(\boldsymbol{x})$ is a free module generated by $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$, $e_2 \wedge e_3$, $e_2 \wedge e_4$, $e_3 \wedge e_4$ with rank 6. Further, $K_p(\boldsymbol{x})$ is in fact isomorphic to the p-th exterior product $\Lambda^p R^n$ of R^n, so $K(\boldsymbol{x})$ is often called the **exterior algebra complex**.

Proposition 11.1.4. $K(\boldsymbol{x})$ is indeed a chain complex and $K_p(\boldsymbol{x})$ is a free R-module with rank $\binom{n}{p}$.

Example 11.1.5. As an example, when $n = 2$ and $\boldsymbol{x} = (x_1, x_2)$, $K(\boldsymbol{x})$ is the total complex

$$0 \to R \cdot (e_1 \wedge e_2) \xrightarrow{d_2} R \cdot e_1 \oplus R \cdot e_2 \xrightarrow{d_1} R \to 0,$$

where $d_2 = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$ and $d_1 = (x_1, x_2)$. Note that indeed $d_1 \circ d_2 = 0$.

11.2. Koszul (Co)homology

Definition 11.2.1. For an R-module A, we define the **Koszul homology** and **Koszul cohomology** to be

$$H_q(\boldsymbol{x}, A) = H_q(K(\boldsymbol{x}) \otimes_R A),$$
$$H^q(\boldsymbol{x}, A) = H^q(\mathrm{Hom}_R(K(\boldsymbol{x}), A)).$$

Proposition 11.2.2. $\{H_q(\boldsymbol{x}, -)\}$ is a homological δ-functor and $\{H^q(\boldsymbol{x}, -)\}$ is a cohomological δ-functor with

$$H_0(\boldsymbol{x}, A) = A/\boldsymbol{x}A, \quad H^0(\boldsymbol{x}, A) = \mathrm{Hom}_R(R/\boldsymbol{x}R, A) = \{a \in A : x_i a = 0 \ \text{ for all } i\}.$$

Proof. Each $K_p(\boldsymbol{x})$ is free and hence flat and projective, so $\big(K_p(\boldsymbol{x}) \otimes_R -\big)$ and $\mathrm{Hom}_R\big(K_p(\boldsymbol{x}), -\big)$ are both exact functors. For any short exact sequence $0 \to A \to B \to C \to 0$, we thus have a short exact sequence of chain complexes,

$$0 \to K(\boldsymbol{x}) \otimes_R A \to K(\boldsymbol{x}) \otimes_R B \to K(\boldsymbol{x}) \otimes_R C \to 0$$

and a short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}_R\big(K_p(\boldsymbol{x}), A\big) \to \mathrm{Hom}_R\big(K_p(\boldsymbol{x}), B\big) \to \mathrm{Hom}_R\big(K_p(\boldsymbol{x}), C\big) \to 0$$

By Theorem 6.3.2, applying homology and cohomology to them respectively induces two long exact sequences. Notice $K_1(\boldsymbol{x}) \cong R^n$ with generators $\{e_i\}_{1 \le i \le n}$ and $K_0(\boldsymbol{x}) = R$. The differential $K_1(\boldsymbol{x}) \to K_0(\boldsymbol{x})$ sends each e_i to x_i. The rest should follow easily. ∎

Proposition 11.2.3. There are isomorphisms $H_p(\boldsymbol{x}, A) \cong H^{n-p}(\boldsymbol{x}, A)$ for all p.

Lemma 11.2.4 (Künneth Formula for Koszul Complexes). If $C_\bullet$ is a chain complex of R-modules and $x \in R$, there are exact sequences

$$0 \to H_0\big(x, H_q(C)\big) \to H_q(K(x) \otimes_R C) \to H_1\big(x, H_{q-1}(C)\big) \to 0$$

Proof. Again recall Notation 8.3.5, so the middle term means $H_q(\mathrm{Tot}^{\oplus}(K(x) \otimes_R C))$. By definition,

$$[\mathrm{Tot}^{\oplus}(K(x) \otimes_R C)]_n = (K_0(x) \otimes_R C_n) \oplus (K_1(x) \otimes_R C_{n-1}) \cong C_n \oplus C_{n-1},$$

where the differential is given by $(c_n, c_{n-1}) \mapsto (d(c_n) + xc_{n-1}, -d(c_{n-1}))$. Thus we can write a short exact sequence of chain complexes:

$$0 \to C \to \mathrm{Tot}^{\oplus}(K(x) \otimes_R C) \to C[-1] \to 0,$$

which is associated to the long exact sequence:

$$H_{q+1}(C[-1]) \xrightarrow{\partial} H_q(C) \to H_q(K(x) \otimes C) \to H_q(C[-1]) \xrightarrow{\partial} H_q(C),$$

where $H_{q+1}(C[-1]) = H_q(C)$ and $H_q(C[-1]) = H_{q-1}(C)$. By Theorem 6.3.2, we can find that the connecting homomorphism ∂ is multiplication by x. Now we have

$$H_q(C) \xrightarrow{x} H_q(C) \to H_q(K(x) \otimes C) \to H_{q-1}(C) \xrightarrow{x} H_{q-1}(C),$$

which, by Proposition 3.5.8, leads to the short exact sequence

$$0 \to \mathrm{Coker}\Big(H_q(C) \xrightarrow{x} H_q(C)\Big) \to H_q(K(x) \otimes C) \to \mathrm{Ker}\Big(H_{q-1}(C) \xrightarrow{x} H_{q-1}(C)\Big) \to 0.$$

Now since

$$H_q(C) \xrightarrow{x} H_q(C) = \Big(R \xrightarrow{x} R\Big) \otimes_R H_q(C),$$

we find

$$\mathrm{Coker}\Big(H_q(C) \xrightarrow{x} H_q(C)\Big) = H_0\big(x, H_q(C)\big) \quad \text{and} \quad \mathrm{Ker}\Big(H_q(C) \xrightarrow{x} H_q(C)\Big) = H_1\big(x, H_q(C)\big),$$

and the result follows. ∎

Now recall that if A is an R-module and $r \in R$, then r is a **zero-divisor** on A if there exists non-zero $a \in A$ such that $ra = 0$. Therefore, r is a **non-zero-divisor** on A if and only if the multiplication $A \xrightarrow{r} A$ is injective.

Definition 11.2.5. If A is an R-module, a **regular sequence** on A is a sequence of elements $(x_1, \ldots, x_n)$ where each $x_i \in R$ such that x_1 is a non-zero-divisor on A and each x_i is a non-zero-divisor on $A/(x_1, \ldots, x_{i-1})A$.

Lemma 11.2.6. Let A be an R-module. If x is a non-zero-divisor on A, then $H_1(x, A) = 0$.

Proof. $K(x) \otimes_R A$ is the chain complex $0 \to A \xrightarrow{x} A \to 0$. If x is a non-zero-divisor on A, then $H_1(x, A) = \operatorname{Ker} x = 0$. ∎

Corollary 11.2.7. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ is a regular sequence on an R-module A, then $H_q(\boldsymbol{x}, A) = 0$ for $q > 0$.

Proof. By induction on n. The base case for $n = 1$ is given in Lemma 11.2.6. Let $x = x_n$ and $\boldsymbol{y} = (x_1, \ldots, x_{n-1})$, then $K(\boldsymbol{x}) = \operatorname{Tot}^{\oplus}(K(x) \otimes_R K(\boldsymbol{y}))$. By Lemma 11.2.4 (letting $C = K(\boldsymbol{y}) \otimes_R A$), we have a short exact sequence

$$0 \to H_0\big(x, H_q(\boldsymbol{y}, A)\big) \to H_q(\boldsymbol{x}, A) \to H_1\big(x, H_{q-1}(\boldsymbol{y}, A)\big) \to 0. \tag{8}$$

For $q \geq 2$, the flanking terms of Short Exact Sequence 8 are both 0 by induction and hence $H_q(\boldsymbol{x}, A) = 0$. For $q = 1$, by induction the left term of Short Exact Sequence 8 is 0, so we get

$$H_1(\boldsymbol{x}, A) \cong H_1(x, H_0(\boldsymbol{y}, A)) = H_1(x, A/(x_1, \ldots, x_{n-1})A) = 0$$

by Proposition 11.2.2 and Lemma 11.2.6, since x is a non-zero-divisor on $A/(x_1, \ldots, x_{n-1})A$. ∎

Corollary 11.2.8 (Koszul resolution). If $\boldsymbol{x} = (x_1, \ldots, x_n)$ is a regular sequence on R (viewed as an R-module), then $K(\boldsymbol{x})$ is a free resolution of R/I, where $I = (x_1, \ldots, x_n)R$.

Proof. Notice that $H_q(\boldsymbol{x}, R) = H_q(K(\boldsymbol{x}) \otimes_R R) \cong H_q(K(\boldsymbol{x}))$. When $q \geq 1$, by Corollary 11.2.7, $H_q(K(\boldsymbol{x})) = 0$. When $q = 0$, $H_0(K(\boldsymbol{x})) = R/\boldsymbol{x}R = R/I$ by Proposition 11.2.2. This indicates that

$$\ldots \to K_2(\boldsymbol{x}) \to K_1(\boldsymbol{x}) \to K_0(\boldsymbol{x}) \to R/I \to 0$$

is exact everywhere. Thus $K(\boldsymbol{x})$ is a free resolution of R/I. ∎

Corollary 11.2.9. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ is a regular sequence on R, $I = (x_1, \ldots, x_n)R$ and B is an R-module, then

$$H_p(\boldsymbol{x}, B) = \operatorname{Tor}_p^R(R/I, B),$$
$$H^p(\boldsymbol{x}, B) = \operatorname{Ext}_p^R(R/I, B).$$

Proof. This follows from the Koszul resolution of R/I and the definition (or the balancing) of Ext and Tor. ∎

Example 11.2.10. Let k be a field, $R = k[x, y]$ and $I = (x, y)R$. Then $k \cong R/I$ and has Koszul resolution

$$0 \to R \xrightarrow{\binom{y}{-x}} R^2 \xrightarrow{(x,y)} R \to k \to 0.$$

To calculate $\mathrm{Tor}_*^R(k,k)$, we could simply use the definition of Tor: delete k from the sequence, apply $(k \otimes_R -)$, and we get

$$0 \to k \xrightarrow{0} k^2 \xrightarrow{0} k \to 0,$$

where the differentials vanish since x and y are modded out in k. Take the homology of the above sequence and we see

$$\mathrm{Tor}_*^R(k,k) \cong \begin{cases} k, & * = 0, 2, \\ k^2, & * = 1, \\ 0, & \text{otherwise.} \end{cases}$$

12. Ext and Extensions

12.1. Extensions

Definition 12.1.1. Let A and B be R-modules. An **extension** of A by B is a short exact sequence

$$0 \to B \to X \to A \to 0.$$

Two extensions are **equivalent** if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\cong} & & \downarrow{=} & & \\ 0 & \longrightarrow & B & \longrightarrow & X' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

This is an equivalence relation. We denote $e(A, B)$ as the equivalence classes of extensions of A by B.

An extension is **split** if it is equivalent to

$$0 \to B \to A \oplus B \to A \to 0.$$

Lemma 12.1.2. If $\mathrm{Ext}^1(A, B) = 0$, then every extension of A by B is split.

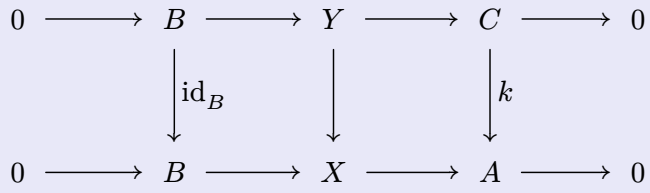
Proof. We look at the long exact sequence of $\mathrm{Ext}^*(A, -)$:

$$\mathrm{Hom}(A, X) \to \mathrm{Hom}(A, A) \xrightarrow{\partial} \mathrm{Ext}^1(A, B) = 0.$$

The first arrow is a surjection, so $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ can always lift to a splitting $\sigma : A \to X$. (It is helpful to recall the proof of Proposition 3.6.2.) ∎

From the above proof, we also see that $\partial(\mathrm{id}_A) \in \mathrm{Ext}^1(A, B)$ is the obstruction to the extension of A by B being split: the extension is split if and only if id_A lifts to some element in $\mathrm{Hom}(A, X)$, if and only if $0 = \partial(\mathrm{id}_A)$.

Lemma 12.1.3. Let $0 \to B \to X \to A \to 0$ be an extension of A by B, and let $k : C \to A$. Then there exists an extension $0 \to B \to Y \to C \to 0$ of C by B, unique up to extension equivalence, such that the following diagram commutes:



Proof. (Rotman 2009, Lemma 7.29). ∎

Theorem 12.1.4. Given R-modules A and B, the map

$$\begin{aligned} \Theta : e(A,B) &\to \operatorname{Ext}^1_R(A,B) \\ [0 \to B \to X \to A \to 0] &\mapsto \partial(\mathrm{id}_A) \end{aligned}$$

is a bijection, and split extensions correspond to $0 \in \operatorname{Ext}^1_R(A,B)$.

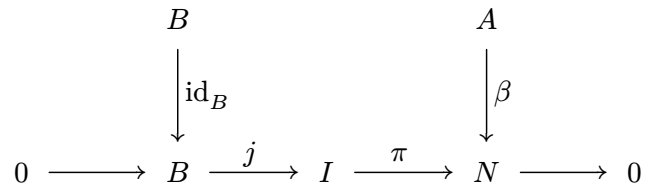
Proof sketch. We first show that Θ is surjective. Let $x \in \operatorname{Ext}^1_R(A,B)$, and we want to construct some extension $\xi = (0 \to B \to X \to A \to 0)$ such that $\Theta(\xi) = x$. Since R-**Mod** has enough injectives, we can find a short exact sequence

$$0 \to B \xrightarrow{j} I \xrightarrow{\pi} N \to 0,$$

where I is injective and $\pi = \operatorname{coker}(j)$. Since I is injective, we have $\operatorname{Ext}^1_R(A,I) = 0$ by Proposition 8.1.3, so the long exact sequence of Ext gives an exact sequence

$$0 \to \operatorname{Hom}_R(A,B) \to \operatorname{Hom}_R(A,I) \to \operatorname{Hom}_R(A,N) \xrightarrow{\delta'} \operatorname{Ext}^1_R(A,B) \to 0.$$

Now δ' is surjective, so $x \in \operatorname{Ext}^1_R(A,B)$ can be lifted to some $\beta \in \operatorname{Hom}_R(A,N)$ such that $\delta'(\beta) = x$. Now we have the following:



We now apply Lemma 12.1.3 to find a commutative diagram with exact rows that completes the above diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_B} & & \downarrow & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & B & \xrightarrow{j} & I & \xrightarrow{\pi} & N & \longrightarrow & 0 \end{array}$$

where the top row is the extension ξ we claim to have $\Theta(\xi) = x$. To prove that it is indeed the case, notice by naturality of the long exact sequence of Ext, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(A, A) & \xrightarrow{\delta} & \mathrm{Ext}^1_R(A, B) \\ \downarrow{\scriptstyle \beta \circ -} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{Hom}_R(A, N) & \xrightarrow{\delta'} & \mathrm{Ext}^1_R(A, B) \end{array}$$

from which we see

$$\Theta(\xi) = \delta(\mathrm{id}_A) = \delta'((\beta \circ -)(\mathrm{id}_A)) = \delta'(\beta) = x.$$

Thus we have shown Θ is surjective.

Now again by Lemma 12.1.3, the extension ξ we have constructed is unique up to equivalence, so we have effectively constructed a well-defined map

$$\Phi : \mathrm{Ext}^1_R(A, B) \to e(A, B)$$

with $\Theta(\Phi(x)) = x$ (if we can show that Φ is independent of the choices of I and β).

The rest of the proof is to show that $\Phi(\Theta([\xi])) = [\xi]$ for any extension class $[\xi] \in e(A, B)$. ■

12.2. Baer Sum

When a set X has a bijection with the underlying map of a group G, then in general X can be also equipped with a group structure. We are therefore interested in characterising the group structure on $e(A, B)$, in view of its bijection with the group $\mathrm{Ext}^1(A, B)$. The natural addition operation on $e(A, B)$ was first explicitly ascertained by R. Baer.

Definition 12.2.1. Let

$$\xi_1 : 0 \to B \xrightarrow{i_1} X_1 \xrightarrow{\pi_1} A \to 0, \quad \xi_2 : 0 \to B \xrightarrow{i_2} X_2 \xrightarrow{\pi_2} A \to 0$$

be extensions of A by B. Let

$$X'' = X_1 \times_A X_2 = \{(x_1, x_2) \in X_1 \times X_2 : \pi_1(x_1) = \pi_2(x_2)\}$$

and let

$$Y = \frac{X''}{\{(i_1(b), -i_2(b)) : b \in B\}}.$$

Then the sequence

$$0 \to B \xrightarrow{i} Y \xrightarrow{\pi} A \to 0$$

is called the **Baer sum** of ξ and ξ', where we have maps

$$\begin{aligned} i : B &\to Y \\ b &\mapsto (i_1(b), 0) \end{aligned}$$

and

$$\begin{aligned} \pi : Y &\to A \\ (x_1, x_2) &\mapsto \pi_1(x_1) + \pi_2(x_2). \end{aligned}$$

Lemma 12.2.2. The Baer sum is a well-defined extension of A by B.

Lemma 12.2.3. The set of equivalence classes of extensions of A by B is an abelian group under the Baer sum, and the map Θ is an isomorphism of abelian groups.

12.3. Yoneda Ext Groups

Using extensions of A by B, we can define $\mathrm{Ext}^n(A, B)$ in any abelian category, not necessarily with enough projectives or injectives. We call this the Yoneda Ext group.

Definition 12.3.1. We define the Yoneda $\mathrm{Ext}^n(A, B)$ to be the equivalence classes of exact sequences

$$\xi : 0 \to B \to X_n \to \ldots \to X_1 \to A \to 0$$

under the equivalence relation generated by $\xi \sim \xi'$ if there is a diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \ldots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & B & \longrightarrow & X_n' & \longrightarrow & \ldots & \longrightarrow & X_1' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Note that the arrows $X_i \to X_i'$ do not have to be isomorphisms. At first glance, this seems different to our previous definition of equivalence for extensions of A by B. However, by the Five Lemma, when $n = 1$, the morphism $X_1 \to X_1'$ is necessarily an isomorphism.

Definition 12.3.2. We again define a notion of a Baer sum. Let ξ and ξ' be representatives of elements of $\mathrm{Ext}^n(A, B)$. Let X_1'' be the pullback of

$$\begin{array}{ccc} & & X_1 \\ & & \downarrow \\ X_1' & \longrightarrow & A \end{array}$$

and let X_n'' be the pushout of

$$\begin{array}{ccc} B & \longrightarrow & X_n \\ \downarrow & & \\ X_n' & & \end{array}$$

Let Y_n be the quotient of X_n'' by the antidiagonal. Then the Baer sum is

$$0 \to B \to Y_n \to X_{n-1} \oplus X_{n-1}' \to \ldots \to X_2 \oplus X_2' \to X_1'' \to A \to 0.$$

Suppose that $\mathcal{A}$ has enough projectives and $P_\bullet \to A$ is a projective resolution. Consider the diagram

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \ldots & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & & & & & & & & & & & \downarrow = & & \\ & & 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \ldots & \longrightarrow & X_0 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

By Theorem 6.4.4, there is a chain map from the top row to the bottom row lifting $\mathrm{id} : A \to A$. Setting $M = \mathrm{Ker}\, d_n^{(P)}$ gives a diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & M & \longrightarrow & P_{n-1} & \longrightarrow & \ldots & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow & & & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \ldots & \longrightarrow & X_0 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows.

Proposition 12.3.3. There is a natural isomorphism between Yoneda Ext^n and the standard Ext^n.

13. Group (Co)homology

13.1. Definitions

Definition 13.1.1. Let G be a group. A **(left) G-module** is an abelian group A together with a left group action $\rho : G \times A \to A$, with $\rho(g, a)$ denoted as $g \cdot a$, such that

$$g \cdot (a_1 + a_2) = g \cdot a_1 + g \cdot a_2$$

for all $g \in G$ and $a_1, a_2 \in A$.

A **morphism** $A \to B$ of G-modules (or a **G-map**) is an abelian group homomorphism (i.e., $\mathbb{Z}$-linear map) $\varphi : A \to B$ such that

$$\varphi(g \cdot a) = g \cdot \varphi(a)$$

for all $g \in G$ and $a \in A$.

The category of G-modules is denoted as G-$\mathbf{Mod}$, where we write $\mathrm{Hom}_{G\text{-}\mathbf{Mod}}$ as Hom_G.

Note 13.1.2. Recall that for any group G, the **integral group ring** $\mathbb{Z}G$ consists of formal sums of elements of G with integer coefficients:

$$\sum_{g \in G} f_g g,$$

where $f_g \in \mathbb{Z}$ is non-zero for only fintely many $g \in G$. $\mathbb{Z}G$ is a ring because the product of two elements of $\mathbb{Z}G$ is well-defined.

Lemma 13.1.3. There is an equivalence of categories G-$\mathbf{Mod} \cong \mathbb{Z}G$-$\mathbf{Mod}$.

This implies that G-modules can be seen as a special case of R-modules, so all the homological algebra we have developed applies.

Definition 13.1.4. A G-module is **trivial** if $g \cdot a = a$ for all $g \in G$ and $a \in A$. We define a functor $\mathrm{triv} : \mathbf{Ab} \to G$-$\mathbf{Mod}$ by sending an abelian group A to a trivial G-module A.

Definition 13.1.5. Let $A \in G$-$\mathbf{Mod}$. Then the submodule of **invariants** of A is

$$A^G = \{a \in A : g \cdot a = a \text{ for all } g \in G\}$$

and the module of **coinvariants** of A is

$$A_G = A/\langle g \cdot a - a : g \in G, a \in A\rangle.$$

Lemma 13.1.6. $-^G$ and $-_G$ are functors $G\text{-}\mathbf{Mod} \to \mathbf{Ab}$.

Lemma 13.1.7. We have adjunctions

$$-_G \dashv \mathrm{triv} \dashv -^G.$$

Proof. We first show

$$\mathrm{Hom}_G(\mathrm{triv}(A), B) \cong \mathrm{Hom}_{\mathbf{Ab}}(A, B^G)$$

Take any $f : \mathrm{triv}(A) \to B$, then f is a group homomorphism $A \to B$ such that $f(g \cdot a) = g \cdot f(a)$ for all $g \in G$ and $a \in A$. But $g \cdot a = a$ due to triviality and hence $f(a) = g \cdot f(a)$, i.e. $f(a) \in B^G$ for all a. Then f is equivalent to a group homomorphism $A \to B^G$.

Now we prove

$$\mathrm{Hom}_{\mathbf{Ab}}(A_G, B) \cong \mathrm{Hom}_G(A, \mathrm{triv}(B))$$

Take any $h : A \to \mathrm{triv}(B)$, then h is a group homomorphism $A \to B$ such that for all $g \in G$ and $a \in A$,

$$h(g \cdot a) = g \cdot h(a) = h(a) \Leftrightarrow h(g \cdot a - a) = 0 \Leftrightarrow g \cdot a - a \in \mathrm{Ker}(h)$$

which means h is equivalent to a group homomorphism $A_G \to B$. ∎

Corollary 13.1.8. The functor $-_G : G\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is right exact and the functor $-^G : G\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is left exact.

Lemma 13.1.9. Let A be any G-module and let $\mathbb{Z}$ be the trivial G-module. Then

$$A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} A$$

and

$$A^G \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A).$$

Remark 13.1.10. In other words, $(-_G) = (\mathbb{Z} \otimes_{\mathbb{Z}G} -) = (- \otimes_{\mathbb{Z}G} \mathbb{Z})$ (because the ring $\mathbb{Z}G$ is commutative) and $\left(-^G\right) = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$.

Proof. We observe that the trivial module functor $\mathrm{triv} : \mathbb{Z}\text{-}\mathbf{Mod} \to \mathbb{Z}G\text{-}\mathbf{Mod}$ can be seen as the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)$, where we consider $\mathbb{Z}$ as a $\mathbb{Z}$-$\mathbb{Z}G$ bimodule. By Theorem 4.5.5, $(\mathbb{Z} \otimes_{\mathbb{Z}G} -)$ is its left adjoint, which must agree with its other left adjoint $-_G$. (See an alternative proof in (Monnet and Kremnitzer 2021, Lemma 13.9).)

For the second claim: $A^G \cong \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, A^G) \cong \mathrm{Hom}_G(\mathbb{Z}, A)$. ∎

Definition 13.1.11. Let A be a G-module. We define the **homology groups of G with coefficients in A** as the left derived functors of $\left(-^G\right)$:

$$H_*(G,A) = L_*\left(-^G\right)(A) \cong \operatorname{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},A).$$

By definition, $H_0(G,A) = A_G$.

Similarly, we define the **cohomology groups of G with coefficients in A** as the right derived functors of $(-_G)$:

$$H^*(G,A) = R^*(-_G)(A) \cong \operatorname{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},A).$$

By definition, $H^0(G,A) = A^G$.

Notation 13.1.12. (Weibel 1994) uses the notations $H_*(G;A)$ and $H^*(G;A)$ (with a semicolon instead of a comma).

13.2. First Homology

Note 13.2.1. Recall the **commutator subgroup** of a group G is defined as

$$[G,G] := \langle g^{-1}h^{-1}gh : g,h \in G\rangle$$

and the **abelianisation** of G is $G/[G,G]$.

The aim of this section is to show that $H_1(G,\mathbb{Z}) \cong G/[G,G]$ for any group G.

Definition 13.2.2. The **augmentation ideal** $\mathfrak{J}$ of $\mathbb{Z}G$ is defined as the kernel of the ring map

$$\begin{aligned}\varepsilon : \mathbb{Z}G &\to \mathbb{Z}\\ \sum_{g\in G} f_g g &\mapsto \sum_{g\in G} f_g.\end{aligned}$$

Remark 13.2.3. ε is obviously a surjection, so $\mathbb{Z} \cong \mathbb{Z}G/\mathfrak{J}$. Moreover, ε can be considered as the start of a resolution of $\mathbb{Z}$.

Lemma 13.2.4. $\mathfrak{J}$ is a free $\mathbb{Z}$-module with basis $\{g-1 : g \in G\setminus\{1\}\}$ (where 1 is the group identity of G).

Proof. Simply notice that $\mathbb{Z}G$ as a free $\mathbb{Z}$-module has a basis $\{1\}\cup\{g-1 : g\in G\setminus\{1\}\}$ and that $\varepsilon(g-1) = 0$ for any $g \in G$. ∎

Lemma 13.2.5. For any G-module A, $H_0(G,A) = A_G \cong A/\mathfrak{J}A$.

Proof. $A_G \cong \mathbb{Z}\otimes_{\mathbb{Z}G} A \cong (\mathbb{Z}G/\mathfrak{J})\otimes_{\mathbb{Z}G} A \cong A/\mathfrak{J}A$, where we use the definition of $\mathfrak{J}$ and Example 4.6.2. ∎

Example 13.2.6. Regarding $\mathbb{Z}$, $\mathbb{Z}G$ and $\mathfrak{J}$ as G-modules, we have $H_0(G,\mathbb{Z}) = \mathbb{Z}/\mathfrak{J}\mathbb{Z} = \mathbb{Z}$, $H_0(G,\mathbb{Z}G) = \mathbb{Z}G/\mathfrak{J} \cong \mathbb{Z}$, and $H_0(G,\mathfrak{J}) = \mathfrak{J}/\mathfrak{J}^2$.

Lemma 13.2.7. $\mathfrak{J}/\mathfrak{J}^2 \cong G/[G,G]$.

Note 13.2.8. $\mathfrak{J}^2$ is the free $\mathbb{Z}$-module with basis $\{(g-1)(h-1) : g,h \in G \setminus \{1\}\}$.

Proof. Define map

$$\begin{aligned} \theta : G &\to \mathfrak{J}/\mathfrak{J}^2 \\ g &\mapsto g - 1 + \mathfrak{J}^2. \end{aligned}$$

Take any $a, b \in G$, then we have

$$\theta(ab) = ab - 1 + \mathfrak{J}^2 = ab - 1 - (a-1)(b-1) + \mathfrak{J}^2 = (a-1) + (b-1) + \mathfrak{J}^2 = \theta(a) + \theta(b),$$

so θ is a group homomorphism. Since $\mathfrak{J}/\mathfrak{J}^2$ is abelian, we have

$$\theta(aba^{-1}b^{-1}) = \theta(a) + \theta(b) - \theta(a) - \theta(b) = 0,$$

so $[G,G] \subseteq \operatorname{Ker}\theta$, and θ descends to a homomorphism $\bar{\theta} : G/[G,G] \to \mathfrak{J}/\mathfrak{J}^2$.

Define group homomorphism $\sigma : \mathfrak{J} \to G/[G,G]$ linearly expanded by

$$n(g-1) \mapsto g^n[G,G].$$

Then for $a, b \in G$, we have

$$\sigma((a-1)(b-1)) = \sigma(ab - 1 - (a-1) - (b-1)) = aba^{-1}b^{-1}[G,G] = [G,G].$$

So σ descends to a homomorphism $\bar{\sigma} : \mathfrak{J}/\mathfrak{J}^2 \to G/[G,G]$. The result thus follows from the obvious fact that $\bar{\sigma}$ and $\bar{\theta}$ are mutual inverses. ∎

Theorem 13.2.9. $H_1(G,\mathbb{Z}) \cong G/[G,G]$.

Proof. We have a short exact sequence

$$0 \to \mathfrak{J} \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

of G-modules, where $\mathbb{Z}$ is viewed as a trivial G-module. The long exact sequence of $\operatorname{Tor}_*^{\mathbb{Z}G}$ gives

$$H_1(G,\mathbb{Z}G) \to H_1(G,\mathbb{Z}) \to \mathfrak{J}_G \to (\mathbb{Z}G)_G \xrightarrow{\varepsilon_*} \mathbb{Z}_G \to 0.$$

Since $\mathbb{Z}G$ is a projective and thus flat $\mathbb{Z}G$-module, we have $H_1(G,\mathbb{Z}G) = 0$ by Proposition 10.1.2. Notice that $\mathbb{Z}_G = \mathbb{Z}$ and $(\mathbb{Z}G)_G \cong \mathbb{Z}$ by Example 13.2.6. Since ε_* is a surjection, we see ε_* must be an isomorphism $\mathbb{Z} \to \mathbb{Z}$. So we have $H_1(G,\mathbb{Z}) \cong \mathfrak{J}_G = \mathfrak{J}/\mathfrak{J}^2 \cong G/[G,G]$. ∎

13.3. Norm Element

Definition 13.3.1. Let G be a finite group. The **norm element** of $\mathbb{Z}G$ is

$$N = \sum_{g \in G} g \in \mathbb{Z}G.$$

Notation 13.3.2. Somehow the convention here is to use a capital letter N for a *group element*, not a group.

Lemma 13.3.3. N is a central element of $\mathbb{Z}G$ and $N \in (\mathbb{Z}G)^G$.

Proof. For every $h \in G$, we have $hN = \sum_g hg$, but left multiplication by h is nothing but a permutation of G (recall Cayley's Theorem), so $hN = \sum_{g'} g' = N$ by reindexing. Similarly, $Nh = N$. ■

Lemma 13.3.4. The subgroup $H^0(G, \mathbb{Z}G) = (\mathbb{Z}G)^G$ is the two-sided ideal $\mathbb{Z} \cdot N$ of $\mathbb{Z}G$ generated by N, and is thus isomorphic to $\mathbb{Z}$.

Proof. Take $a = \sum_{g \in G} n_g g \in (\mathbb{Z}G)^G$. Then for any $h \in G$, $a = ha = \sum_{g \in G} n_g (hg)$. The coefficient for g in ha is $n_{h^{-1}g}$. Thus, for any $g, h \in G$, we have $n_g = n_{h^{-1}g}$, which shows that all n_g are the same. Hence $a = nN$ for some $n \in \mathbb{Z}$. ■

Lemma 13.3.5. When group G is finite,

$$\mathfrak{J} = \mathrm{Ker}\Big(\mathbb{Z}G \xrightarrow{N} \mathbb{Z}G\Big) = \{a \in \mathbb{Z}G : Na = 0\}$$
$$\mathbb{Z} \cdot N = \mathrm{Im}\Big(\mathbb{Z}G \xrightarrow{N} \mathbb{Z}G\Big).$$

Proof. Take $a = \sum_{g \in G} n_g g \in \mathbb{Z}G$ and write $N = \sum_{h \in G} h$. We have

$$Na = \left(\sum_{h \in G} h\right)\left(\sum_{g \in G} n_g g\right) = \sum_{h \in G}\sum_{g \in G} n_g (hg) = \sum_{g' \in G}\sum_{g \in G} n_g g'$$
$$= \sum_{g' \in G}\left(\sum_{g \in G} n_g\right) g' = \left(\sum_{g \in G} n_g\right)\left(\sum_{g' \in G} g'\right) = \left(\sum_{g \in G} n_g\right) N$$

Therefore $Na = 0$ if and only if $\sum_{g \in G} n_g = 0$, if and only if $a \in \mathfrak{J}$. The image of $\mathbb{Z}G \xrightarrow{N} \mathbb{Z}G$ is also clear from above, since $\left(\sum_{g \in G} n_g\right)$ can take all values in $\mathbb{Z}$. ■

Corollary 13.3.6. For every finite group G, there is a short exact sequence

$$0 \to \mathfrak{J} \to \mathbb{Z}G \xrightarrow{N} \mathbb{Z} \cdot N \to 0.$$

13.4. Finite Cyclic Groups

Let $C_m = \langle \sigma : \sigma^m = 1 \rangle$ be the cyclic group of order m generated by σ. Then the norm element of C_m is

$$N = \sum_{i=0}^{m-1} \sigma^i = 1 + \sigma + ... + \sigma^{m-1}$$

We observe

$$0 = \sigma^m - 1 = (\sigma - 1)N$$

Remark 13.4.1. The group ring of C_m can be also viewed as $\mathbb{Z}[\sigma]/(\sigma^m - 1)$.

Lemma 13.4.2. There is a short exact sequence

$$0 \to \mathbb{Z} \cdot N \to \mathbb{Z}C_m \overset{\sigma-1}{\to} \mathfrak{J} \to 0.$$

Proof. We calculate the image and kernel of the map $\mathbb{Z}C_m \overset{\sigma-1}{\to} \mathbb{Z}C_m$. Take $a = \sum_{j=0}^{m-1} n_j \sigma^j \in \mathbb{Z}C_m$. Then setting $n_{-1} = n_{m-1}$, we have

$$(\sigma - 1)a = \sum_{j=0}^{m-1} n_j \sigma^{j+1} - \sum_{j=0}^{m-1} n_j \sigma^j = \sum_{j=0}^{m-1} (n_{j-1} - n_j)\sigma^j.$$

Since

$$\varepsilon((\sigma - 1)a) = \sum_{j=0}^{m-1} (n_{j-1} - n_j) = 0,$$

we see that $(\sigma - 1)a \in \mathfrak{J}$. On the other hand, for any $b = \sum_{k=0}^{m-1} f_k \sigma^k \in \mathfrak{J}$ such that $\sum_{k=0}^{m-1} f_k = 0$, we can find a such that $n_j = -\sum_{k=0}^{j} f_k$ for $j = 0, ..., m-1$ (notice that $n_{m-1} = n_{-1} = 0$) so that $n_{j-1} - n_j = f_j$, or $(\sigma - 1)a = b$. This shows that $\mathrm{Im}\left(\mathbb{Z}C_m \overset{\sigma-1}{\to} \mathbb{Z}C_m\right) = \mathfrak{J}$.

For the kernel, $(\sigma - 1)a = 0$ if and only if $n_{j-1} = n_j$ for all j, if and only if all n_j are equal, if and only if $a \in \mathbb{Z} \cdot N$. ∎

Lemma 13.4.3. The chain complex

$$... \to \mathbb{Z}C_m \overset{\sigma-1}{\to} \mathbb{Z}C_m \overset{N}{\to} \mathbb{Z}C_m \overset{\sigma-1}{\to} \mathbb{Z}C_m \overset{\varepsilon}{\to} \mathbb{Z} \to 0$$

is a projective resolution for $\mathbb{Z}$ as a $\mathbb{Z}C_m$-module.

Proof. This is obtained by splicing the sequences from Corollary 13.3.6 and Lemma 13.4.2 together. ∎

Theorem 13.4.4. Let A be a G-module, where $G = C_m$. Then

$$H_n(C_m, A) = \begin{cases} A/(\sigma-1)A & \text{if } n = 0, \\ A^G/NA & \text{if } n = 1, 3, 5, ..., \\ \{a \in A : Na = 0\}/(\sigma-1)A & \text{if } n = 2, 4, 6, ... \end{cases}$$

$$H^n(C_m, A) = \begin{cases} A^G & \text{if } n = 0, \\ \{a \in A : Na = 0\}/(\sigma-1)A & \text{if } n = 1, 3, 5, ..., \\ A^G/NA & \text{if } n = 2, 4, 6, ... \end{cases}$$

Corollary 13.4.5. We have

$$H_n(C_m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/m & \text{if } n \geq 1 \text{ is odd}, \\ 0 & \text{else.} \end{cases}$$

$$H^n(C_m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/m & \text{if } n \geq 2 \text{ is even}, \\ 0 & \text{else.} \end{cases}$$

13.5. Free Groups

(Weibel 1994, p. 169).

Proposition 13.5.1. Let G be the free group on the set X, and consider the augmentation ideal $\mathfrak{J}$ of $\mathbb{Z}G$. Then $\mathfrak{J}$ is a free $\mathbb{Z}G$-module with basis the set $X - 1 = \{x - 1 : x \in X\}$.

Proof. Every $g \in G$ may be written uniquely as a reduced word in the symbols $\{x, x^{-1} : x \in X\}$; write $G(x)$ (resp. $G(x^{-1})$) for the subset of all $g \in G$ ending in the symbol x (resp. in x^{-1}) so that

$$G - \{1\} = \{G(x)\}_{x \in X} \sqcup \{G(x^{-1})\}_{x \in X}.$$

By Lemma 13.2.4, $\mathfrak{J}$ is a free abelian group with $\mathbb{Z}$-basis $\{g - 1 : g \in G, g \neq 1\}$. Now we claim that the $\mathbb{Z}$-basis $\{g - 1 : g \in G, g \neq 1\}$ can be uniquely rewritten in terms of the set $\{g(x - 1) : g \in G, x \in X\}$. We prove this by induction on word length of g. When the word length is 1, either $g = x$ or $g = x^{-1}$ for some $x \in X$, so the claim is trivial. When the word length is n, then we can write either $g = g'x$ (if $g \in G(x)$) or $g = g'x^{-1}$ (if $g \in G(x^{-1})$), where the word length of g' is $(n - 1)$. In the first case,

$$g - 1 = g'x - 1 = g'(x - 1) + (g' - 1),$$

and in the second case,

$$g - 1 = g'x^{-1} - 1 = -(g'x^{-1})(x - 1) + (g' - 1).$$

Hence in both cases, the claim follows from the induction hypothesis. We can similarly prove that we can uniquely rewrite $\{g(x - 1)\}$ in terms of $\{g - 1 : g \neq 1\}$. Therefore $\{g(x - 1) : g \in G, x \in X\}$ is another $\mathbb{Z}$-basis of $\mathfrak{J}$, and $X - 1 = \{x - 1 : x \in X\}$ is a $\mathbb{Z}G$-basis. ∎

Corollary 13.5.2. If G is a free group on X, then $\mathbb{Z}$ has the free resolution

$$0 \to \mathfrak{J} \to \mathbb{Z}G \to \mathbb{Z} \to 0.$$

Consequently, $H_n(G, A) = H^n(G, A) = 0$ for $n \neq 0, 1$.

Moreover, when $A = \mathbb{Z}$,

$$H_0(G, \mathbb{Z}) \cong H^0(G, \mathbb{Z}) \cong \mathbb{Z}$$

$$H_1(G, \mathbb{Z}) \cong \bigoplus_{x \in X} \mathbb{Z}$$

$$H^1(G, \mathbb{Z}) \cong \prod_{x \in X} \mathbb{Z}$$

Proof. $H_*(G, A)$ is the homology of $0 \to \mathfrak{J} \otimes_{\mathbb{Z}G} A \to A \to 0$, and $H^*(G, A)$ is the cohomology of $0 \to A \to \mathrm{Hom}_G(\mathfrak{J}, A) \to 0$. For $A = \mathbb{Z}$, $H_0(G, \mathbb{Z})$ and $H^0(G, \mathbb{Z})$ come from Example 13.2.6 and Lemma 13.3.4, respectively. $H_1(G, \mathbb{Z}) \cong G/[G, G]$ by Theorem 13.2.9, where $G/[G, G]$ is the free abelian group over X. We finally see that the differential $\mathbb{Z} \to \mathrm{Hom}_G(\mathfrak{J}, \mathbb{Z})$ must be zero, hence $H^1(G, \mathbb{Z}) = \mathrm{Hom}_G(\mathfrak{J}, \mathbb{Z}) \cong \prod_{x \in X} \mathbb{Z}$. ∎

13.6. Derivations

Definition 13.6.1. Let G be a group and A be a left G-module. A **derivation** of G in A is a set map $D : G \to A$ with

$$D(gh) = gD(h) + D(g)$$

for all $g, h \in G$. Write $\mathrm{Der}(G, A)$ for the set of derivations of G in A.

Remark 13.6.2. In general, if R is a ring and A is an R-R-bimodule, a derivation of R in A is an abelian group homomorphism $D : R \to A$ such that

$$D(rs) = rD(s) + D(r)s.$$

Here for $R = \mathbb{Z}G$, we have $D(g)h = D(g)$ because we are viewing A as a $\mathbb{Z}G$-$\mathbb{Z}G$-bimodule with trivial G-action on the right.

Lemma 13.6.3. $\mathrm{Der}(G, A)$ is an abelian group under pointwise addition.

Definition 13.6.4. For $a \in A$, let $D_a : G \to A$ be the map $D_a(g) = ga - a$. A derivation of the form D_a is a **principal derivation**. Write $\mathrm{PDer}(G, A)$ for the set of principal derivations of G in A.

Lemma 13.6.5. $D_a + D_b = D_{a+b}$ and $\mathrm{PDer}(G, A)$ is a subgroup of $\mathrm{Der}(G, A)$.

Lemma 13.6.6. $\mathrm{PDer}(G, A) \cong A/A^G$.

Definition 13.6.7. Let $\varphi : \mathfrak{J} \to A$ be a G-map. Define $D_\varphi : G \to A$ by $D_\varphi(g) = \varphi(g-1)$.

Lemma 13.6.8. The map $\varphi \mapsto D_\varphi : \mathrm{Hom}_G(\mathfrak{J}, A) \to \mathrm{Der}(G, A)$ is a natural isomorphism of abelian groups.

Proof. (Weibel 1994, Lemma 6.4.4). First we show that $D_\varphi : G \to A$ is indeed a derivation:

$$D_\varphi(gh) = \varphi(gh-1) = \varphi(gh-g) + \varphi(g-1) = gD_\varphi(h) + D_\varphi(g)$$

The map $\varphi \mapsto D_\varphi$ is obviously a natural group homomorphism, so it remains to verify that it is an isomorphism.

Suppose $D_\varphi = 0$, i.e., $D_\varphi(g) = \varphi(g-1) = 0$ for all $g \in G$. Since $\{g-1 : g \neq 1\}$ forms a basis for $\mathfrak{J}$, we see that $\varphi = 0$. Hence the map $\varphi \mapsto D_\varphi$ is an injection.

Take any $D \in \mathrm{Der}(G, A)$. Define $\varphi : \mathfrak{J} \to A$ by $\varphi(g-1) = D(g)$ for all $g \neq 1$. This extends to an abelian group homomorphism since $\{g-1 : g \neq 1\}$ forms a basis of $\mathfrak{J}$. It is easy to show that φ is a G-map and $D_\varphi = D$, so the map $\varphi \mapsto D_\varphi$ is also a surjection. ∎

Theorem 13.6.9. $H^1(G, A) \cong \mathrm{Der}(G, A)/\ \mathrm{PDer}(G, A)$.

Proof. (Weibel 1994, Theorem 6.4.5). The short exact sequence

$$0 \to \mathfrak{J} \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

of $\mathbb{Z}G$-modules gives a long exact sequence beginning with

$$0 \to \mathrm{Hom}_G(\mathbb{Z}, A) \to \mathrm{Hom}_G(\mathbb{Z}G, A) \to \mathrm{Hom}_G(\mathfrak{J}, A) \to \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, A) \to \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G, A)$$

which reduces to

$$0 \to A^G \to A \to \mathrm{Der}(G, A) \to H^1(G, A) \to 0$$

The result then follows from Lemma 13.6.6. ∎

Corollary 13.6.10. Let A be a trivial G-module. Then

$$H^1(G, A) \cong \mathrm{Der}(G, A) \cong \mathrm{Hom}_{\mathbf{Grp}}(G, A).$$

13.7. Bar Complexes

Throughout this section, $\mathbb{Z}$ is a trivial G-module.

Definition 13.7.1. The **unnormalised bar complex** is the chain complex

$$... \to B_2^u \to B_1^u \to B_0^u \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

with $B_0^u = \mathbb{Z}G$ and B_n^u is the free $\mathbb{Z}G$-module on the set of all symbols $[g_1 \otimes ... \otimes g_n]$ with $g_i \in G$ for $n \geq 1$. The differential $d : B_n^u \to B_{n-1}^u$ is given by

$$d = \sum_{i=0}^{n} (-1)^i d_i,$$

where

$$\begin{aligned} d_0([g_1 \otimes ... \otimes g_n]) &= g_1[g_2 \otimes ... \otimes g_n] \\ d_i([g_1 \otimes ... \otimes g_n]) &= [g_1 \otimes ... \otimes g_i g_{i+1} \otimes ... \otimes g_n] \quad \text{for } 1 \leq i \leq n-1 \\ d_n([g_1 \otimes ... \otimes g_n]) &= [g_1 \otimes ... \otimes g_{n-1}]. \end{aligned}$$

Definition 13.7.2. The **normalised bar complex** is

$$... \to B_2 \to B_1 \to B_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where $B_0 = \mathbb{Z}G$, and for $n \geq 1$, the group B_n is the free $\mathbb{Z}G$-module on the set of all symbols $[g_1|...|g_n]$ with $g_i \in G \setminus \{1\}$. The differential $d : B_n \to B_{n-1}$ is $d = \sum_{i=0}^{n} (-1)^i d_i$, where

$$\begin{aligned} d_0([g_1|...|g_n]) &= g_1[g_2|...|g_n] \\ d_i([g_1|...|g_n]) &= [g_1|...|g_i g_{i+1}|...|g_n] \quad \text{for } 1 \leq i \leq n-1 \\ d_n([g_1|...|g_n]) &= [g_1|...|g_{n-1}] \end{aligned}$$

We write [] for $1 \in B_0 = \mathbb{Z}G$. If any of the g_i is 1 , we write $[...|g_i|...]$ for $0 \in B_n$.

Example 13.7.3. We have

$$\begin{aligned} d([g|h]) &= g[h] - [gh] + [g], \\ d([f|g|h]) &= f[g|h] - [fg|h] + [f|gh] - [f|g]. \end{aligned}$$

Theorem 13.7.4. The normalised and unnormalised bar complexes are free resolutions of $\mathbb{Z}$ as a $\mathbb{Z}G$-module.

Proof. (Weibel 1994, Theorem 6.5.3). We only give the proof for the normalised bar complexes, as the unnormalised case is the same. By Corollary 6.2.5, we only need to show that there exist abelian group

homomorphisms $s_{-1} : \mathbb{Z} \to B_0$ and $s_n : B_n \to B_{n+1}$ for $n \geq 0$ such that $ds + sd = 1$. The desired construction is given as $s_{-1}(1) = []$ and

$$s_n(g_0[g_1|...|g_n]) = [g_0|g_1|...|g_n]$$

for $n \geq 0$. ■

Corollary 13.7.5. $H^*(G, A)$ is the cohomology of either the chain complex $\mathrm{Hom}_G(B^u_*, A)$ or $\mathrm{Hom}_G(B_*, A)$.

This allows us to give an explicit description of group cohomology.

Definition 13.7.6. Define an n-**cochain** as a function $f : G^n \to A$. An n-cochain φ is **normalised** if $\varphi(g_1, ..., g_n) = 0$ whenever there exists some $g_i = 1$. Define the differential d of an n-cochain φ as an $(n+1)$-cochain $d\varphi$ given by

$$(d\varphi)(g_1, ..., g_{n+1}) = g_1\varphi(g_2, ..., g_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(..., g_i g_{i+1}, ...) + (-1)^{n+1}\varphi(g_1, ..., g_n).$$

If φ is an n-cochain such that $d\varphi = 0$, then φ is an n-**cocycle**. If φ' is an $(n-1)$-cochain, then the n-cochain $d\varphi'$ is an n-**coboundary**. Write $Z^n(G, A)$ and $B^n(G, A)$ for the abelian groups of n-cocycles and n-coboundaries respectively.

From the definition, we see that $\mathrm{Hom}_G(B^u_n, A)$ consists of all n-cochains, while $\mathrm{Hom}_G(B_n, A)$ consists of all normalised n-cochains.

Corollary 13.7.7. $H^n(G, A) = Z^n(G, A)/B^n(G, A)$.

Example 13.7.8. $H^1(G, A) = \mathrm{Der}(G, A)/\ \mathrm{PDer}(G, A)$.

Proof. (Weibel 1994, Example 6.5.6). This is a direct proof of Theorem 13.6.9 using bar resolutions. A 0-cochain is a map $1 \to A$, that is, an element of A. If $a \in A$, then da is the map $G \to A$ sending g to $ga - a$, which is a principal derivation by definition. Therefore, $\varphi \in B^1(G, A)$ if and only if there exists $a \in A$ such that $\varphi = da$, if and only if $\varphi \in \mathrm{PDer}(G, A)$. So $B^1(G, A) = \mathrm{PDer}(G, A)$.

On the other hand, $\varphi \in Z^1(G, A)$ if and only if $d\varphi = 0$, if and only if for all $g, h \in G$,

$$0 = (d\varphi)(g, h) = g\varphi(h) - \varphi(gh) + \varphi(g)$$

if and only if $\varphi \in \mathrm{Der}(G, A)$. Thus $Z^1(G, A) = \mathrm{Der}(G, A)$. ■

13.8. Group Extensions

Definition 13.8.1. Let A be an abelian group and let G be a group. An **extension** of G by A is a short exact sequence

$$0 \to A \to E \xrightarrow{\pi} G \to 1.$$

The extension **splits** if π has a section, i.e., if there is a group homomorphism $s : G \to E$ such that $\pi \circ s = \mathrm{id}_G$. Extensions

$$0 \to A \to E_i \xrightarrow{\pi} G \to 1,$$

for $i = 1, 2$ are **equivalent** if there is a group isomorphism $E_1 \to E_2$ such that the obvious diagram commutes.

Theorem 13.8.2. There is a natural bijection between $H^2(G, A)$ and the equivalence classes of extensions of G by A.

Proof. (Weibel 1994, Classification Theorem 6.6.3). ∎

14. Example: $R = \mathbb{Z}$

Let $A \in \mathbf{Ab}$. Recall that we have the following:

- A is projective if and only if A is free;
- A is injective if and only if A is divisible, if and only if A is a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$, where each p is prime;
- A is flat if and only if A is torsionfree.

We now demonstrate some calculations in the category $\mathbf{Ab}$ using tools developed throughout the course. These may serve as exercises or reference materials.

14.1. Resolutions

Type	Object	Resolution
projective	$\mathbb{Z}/m$	$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0$
injective	$\mathbb{Z}/m$	$0 \to \mathbb{Z}/m \to \mathbb{Q}/\mathbb{Z} \xrightarrow{m} \mathbb{Q}/\mathbb{Z} \to 0$
injective	$\mathbb{Z}$	$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

14.2. Tensor products

$\otimes_{\mathbb{Z}}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Z}/m$	$\mathbb{Q}/\mathbb{Z}$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Z}/m$	$\mathbb{Q}/\mathbb{Z}$
$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	0	0
$\mathbb{Z}/n$	$\mathbb{Z}/n$	0	$\mathbb{Z}/\gcd(m,n)$	0
$\mathbb{Q}/\mathbb{Z}$	$\mathbb{Q}/\mathbb{Z}$	0	0	0

- $A \otimes_{\mathbb{Z}} B \cong B \otimes_{\mathbb{Z}} A$;
- $\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A$;
- $\mathbb{Q} \otimes_{\mathbb{Z}} T \cong 0$ if T is torsion;
- $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} T \cong 0$ if T is torsion;
- $\mathbb{Z}/m \otimes_{\mathbb{Z}} A \cong A/mA$ (recall that $R/I \otimes_R M \cong M/IM$);
- $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

14.3. Tor groups

$\mathrm{Tor}_1^{\mathbb{Z}}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Z}/m$	$\mathbb{Q}/\mathbb{Z}$
$\mathbb{Z}$	0	0	0	0
$\mathbb{Q}$	0	0	0	0
$\mathbb{Z}/n$	0	0	$\mathbb{Z}/\gcd(m,n)$	$\mathbb{Z}/n$
$\mathbb{Q}/\mathbb{Z}$	0	0	$\mathbb{Z}/m$	$\mathbb{Q}/\mathbb{Z}$

- $\mathrm{Tor}_n^{\mathbb{Z}}(A,B) \cong \mathrm{Tor}_n^{\mathbb{Z}}(B,A)$;
- $\mathbb{Z}$ is free and thus flat, so $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},-) = 0$;
- $\mathbb{Q}$ is torsionfree and thus flat, so $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q},-) = 0$;
- $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m, A) \cong \{a \in A \mid ma = 0\}$;

- $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is obtained from the long exact sequence induced by applying $(- \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$ to $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.

14.4. Hom-sets

$\operatorname{Hom}_{\mathbb{Z}}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Z}/m$	$\mathbb{Q}/\mathbb{Z}$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Z}/m$	$\mathbb{Q}/\mathbb{Z}$
$\mathbb{Q}$	0	$\mathbb{Q}$	0	?
$\mathbb{Z}/n$	0	0	$\mathbb{Z}/\gcd(m,n)$	$\mathbb{Z}/n$
$\mathbb{Q}/\mathbb{Z}$	0	0	0	$\hat{\mathbb{Z}}$

Each row represents the first argument in Hom and each column the second. A question mark indicates that the result is beyond the scope of the course.

- $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \cong A$ (in general, $\operatorname{Hom}_R(R, M) \cong M$);
- $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$ by establishing a (fairly easy) bijection;
- $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \cong 0$ because no integer is arbitrarily divisible, e.g. $f(1) = nf\left(\frac{1}{n}\right)$ where $f(1), f\left(\frac{1}{n}\right) \in \mathbb{Z}$, so $f(1)$ is divisible by any n which is impossible unless $f(1) = 0$. Similarly $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/m) \cong 0$;
- $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, A) \cong \{a \in A \mid na = 0\}$ (incidentally, this is isomorphic to $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, A)$);
- $\operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is the **Pontryagin duality** of A.

14.5. Ext groups

$\operatorname{Ext}_1^{\mathbb{Z}}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Z}/m$	$\mathbb{Q}/\mathbb{Z}$
$\mathbb{Z}$	0	0	0	0
$\mathbb{Q}$	?	0	?	0
$\mathbb{Z}/n$	$\mathbb{Z}/n$	0	$\mathbb{Z}/\gcd(m,n)$	0
$\mathbb{Q}/\mathbb{Z}$	?	0	?	0

Each row represents the first argument in Ext and each column the second.

- $\operatorname{Ext}_1^{\mathbb{Z}}(\mathbb{Z}, -) = 0$ as $\mathbb{Z}$ is projective;
- $\operatorname{Ext}_1^{\mathbb{Z}}(-, \mathbb{Q}) = 0$ as $\mathbb{Q}$ is injective;
- $\operatorname{Ext}_1^{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) = 0$ as $\mathbb{Q}/\mathbb{Z}$ is injective;
- $\operatorname{Ext}_1^{\mathbb{Z}}(\mathbb{Z}/n, A) \cong A/nA$ using the projective resolution of $\mathbb{Z}/n$.

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