# Infinite Groups

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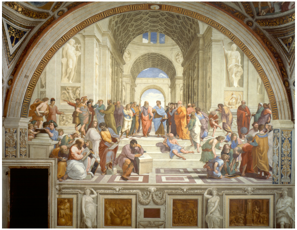
At the entrance to Plato's Academy, an inscription over the door said: Let no one destitute of geometry enter here. The same is written at the entrance to the Mathematical Institute.

William Thurston:
"Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding."



# Course material available on the webpage

- Lecture Notes
- Revision Notes: material from courses in previous years, to be used as a reference only. Not examinable. Occasional reminders in lectures.
- Hand out Notes: expand on some notions introduced in the course, as further reading for students who wish to have a better understanding of the latter. Not examinable.
- Mini-projects Broadening. Feel free to suggest other projects as well, as long as they are related to this course. Assessed by supervisors and Director of Graduate Studies. Assessors may ask about the topics covered in broadening training during Transfer or Confirmation.

# Themes of the course

We study countable infinite groups. Why should we?

- They may appear as groups of transformations preserving a structure and as such encode a lot of information: e.g. for surfaces their mapping class groups.
- Some are closely connected to Number Theory: e.g. the Littlewood conjecture can be reformulated in terms of the geometry of $SL(3, \mathbb{Z})$.
- They have close connections to Post-Quantum Cryptography.

  https://sites.google.com/view/pqa-ihp-2024/computational-group-theory-and-applications-workshop

# Classes of infinite groups that we study

- “Small”: Abelian finitely generated $\subset$ Nilpotent finitely generated $\subset$ Polycyclic $\subset$ Solvable finitely generated.
- “Large”: Free groups $\subset$ Hyperbolic groups.

  Free groups $\subset$ Amalgamated products (in the sense of J.P. Serre).

The families of “small” groups are the object of study of the C2.4 “Infinite Groups” course.

The families of “large” groups are the object of study of the C3.2 “Geometric Group Theory” course in Hilary Term.

# Methods of study I:

(1) Geometric Approach: Make the group $G$ act on an interesting metric space $X$. Deduce algebraic properties of $G$ from

- the geometry of $X$
- the properties of the action of $G$ on $X$

For example, $X$ might be a Hilbert space (in particular $\mathbb{R}^n$) or a differentiable/Riemannian manifold.

Or endow the group itself with a metric, a geometry.

- Easiest way to do it: using Cayley graphs.
- Effective for groups with a finite generating set.
- A recurrent theme: search for a connection between algebraic features of the group and geometric features of its Cayley graphs.

# Methods of study II:

(2) Approximation by finite groups:

- For instance by larger and larger finite quotients. Idea is to take finite quotients $G/N_k$ that become larger and larger. Ideally $\bigcap N_k = \{1\}$.
- This can be done for residually finite groups.

# Methods of study III:

(3) Algorithmic Approach: Design algorithms/construct Turing machines that can find solutions to algebraic questions. This works for groups that can be fully described to a computer via finite data (i.e. finitely presented groups).

Dehn (1912) formulated 3 fundamental problems:

I Word Problem

II Conjugacy Problem

III Isomorphism Problem

# Methods of study IV:

## (4) Linear groups:

We can try to represent infinite groups as groups of matrices.

The groups that can be thus represented are called linear groups.

Not very effective for “large” groups, but very relevant for “small” groups.

## (5) Understanding of subgroups:

- What kind of subgroups?
- Can decompose $G$ into “building blocks”?

- “Small” groups: basic building blocks are abelian; the general groups obtained by iterating semidirect products, more generally short exact sequences;
- “Large” groups: a larger class of building blocks; we iterate amalgamated products and HNN extensions.

# Constructions of groups, old and new

Direct product

The standard approach: take $H_1, H_2$, define operation on $H_1 \times H_2$.

Another approach: given $G$ and $H_1, H_2$ subgroups of $G$, how to decide if $G$ is isomorphic to $H_1 \times H_2$?
Three conditions:

- $H_1, H_2$ both normal subgroups.
- $H_1 \cap H_2 = \{1\}$.
- $H_1 H_2 = G$.

## Generalization: direct sum

Let $X \neq \emptyset$, $\mathcal{G} = \{G_x \mid x \in X\}$ a collection of groups.
Consider

$$Map_f(X, \mathcal{G}) := \{f : X \to \bigsqcup_{x \in X} G_x \mid f(x) \in G_x\,,\ f(x) \neq 1_{G_x}$$

$$\text{for only finitely many } x \in X\}\,.$$

The direct sum $\bigoplus_{x \in X} G_x$ is $Map_f(X, \mathcal{G})$, endowed with the pointwise multiplication:

$$(f \cdot g)(x) = f(x) \cdot g(x)\,,\ \forall x \in X.$$

When $G_x = G$ for all $x \in X$, the direct sum is denoted by $\bigoplus_{x \in X} G$.

# Semidirect product

Given two groups $N$ and $H$ and a group homomorphism $\varphi : H \to \mathrm{Aut}\,(N)$, one can define a new group $G = N \rtimes_\varphi H$ called semidirect product of $N$ and $H$ with respect to $\varphi$:

- As a set, $N \rtimes_\varphi H$ is defined as the cartesian product $N \times H$.
- Binary operation $*$ on $G$ defined by

  $$(n_1, h_1) * (n_2, h_2) = (n_1\varphi(h_1)(n_2), h_1 h_2)\,,\ \forall n_1, n_2 \in N \text{ and } h_1, h_2 \in H\,.$$

- If $\varphi$ is trivial (i.e. has as image $\{\mathrm{id}_N\}$) then $N \rtimes_\varphi H$ is the direct product $N \times H$.

# Semi-direct product 2

Given a group $G$ and two subgroups $H, N$ how to know if $G$ isomorphic to $N \rtimes_{\varphi} H$ for some $\varphi$?

Again three conditions:

- $N$ normal subgroup.
- $N \cap H = \{1\}$.
- $NH = G$.

If the above are satisfied, $G$ isomorphic to $N \rtimes_{\varphi} H$, where $\varphi(h) =$ conjugation by $h$ of every element in $N$.

# A more general notion

An exact sequence is a sequence of groups and group homomorphisms

$$\dots G_{n-1} \xrightarrow{\varphi_{n-1}} G_n \xrightarrow{\varphi_n} G_{n+1} \dots$$

such that $\mathrm{Im}\,\varphi_{n-1} = \ker \varphi_n$ for every $n$.

A short exact sequence is an exact sequence of the form:

$$\{1\} \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow \{1\}\,. \tag{1}$$

In other words, $\varphi$ is an isomorphism from $N$ to a normal subgroup $N' \lhd G$ and $\psi$ defines an isomorphism $G/N' \simeq H$.

If $G$ isomorphic to $N \rtimes_\varphi H$ then we have a short exact sequence as above. The converse is in general not true.

# Semidirect product and short exact sequence

A short exact sequence splits if there exists a homomorphism $\sigma : H \to G$ (called a section) such that

$$\psi \circ \sigma = \mathrm{id}.$$

A split exact sequence determines a decomposition of $G$ as a semidirect product $\varphi(N) \rtimes \sigma(H)$.

Examples

1. The dihedral group $D_{2n}$ is isomorphic to $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$, where $\varphi(1)(k) = n - k$.
2. The infinite dihedral group $D_\infty$ is isomorphic to $\mathbb{Z} \rtimes_\varphi \mathbb{Z}_2$, where $\varphi(1)(k) = -k$.
3. The permutation group $S_n$ is the semidirect product of $A_n$ and $\mathbb{Z}_2 = \{\mathrm{id}, (12)\}$.

# Wreath product

Consider a direct sum $\bigoplus_{x\in H} G$ with index set a group $H$.

There is a natural action of $H$ on the direct sum:

$$\varphi : H \to \mathrm{Aut}\left(\bigoplus_{h\in H} G\right), \varphi(h)f(x) = f(h^{-1}x), \forall x \in H .$$

Thus, we define the semidirect product

$$\left(\bigoplus_{h\in H} G\right) \rtimes_{\varphi} H . \qquad (2)$$

The semidirect product (2) is called the wreath product of $G$ with $H$, and it is denoted by $G \wr H$.

The wreath product $G = \mathbb{Z}_2 \wr \mathbb{Z}$ is called the lamplighter group. Its name comes from the way in which $\varphi(1)$ acts.

The wreath product construction is a source of interesting examples of groups, in particular of solvable groups.