Infinite Groups

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Direct sum

Let $X \neq \emptyset$, $\mathcal{G} = \{G_x \mid x \in X\}$ a collection of groups.
Consider

$$Map_f(X, \mathcal{G}) := \{f : X \to \bigsqcup_{x \in X} G_x \mid f(x) \in G_x \, , \, f(x) \neq 1_{G_x}$$

$$\text{for only finitely many } x \in X\} \, .$$

The direct sum $\bigoplus_{x \in X} G_x$ is $Map_f(X, \mathcal{G})$, endowed with the pointwise multiplication:

$$(f \cdot g)\,(x) = f(x) \cdot g(x) \, , \, \forall x \in X.$$

Semidirect product

Given two groups N and H and a group homomorphism $\varphi : H \to \mathrm{Aut}\,(N)$, one can define a new group $G = N \rtimes_\varphi H$ called semidirect product of N and H with respect to φ:

- As a set, $N \rtimes_\varphi H$ is defined as the cartesian product $N \times H$.
- Binary operation $*$ on G defined by

$$(n_1, h_1) * (n_2, h_2) = (n_1 \varphi(h_1)(n_2), h_1 h_2)\,,\ \forall n_1, n_2 \in N \text{ and } h_1, h_2 \in H\,.$$

- If φ is trivial (i.e. has as image $\{\mathrm{id}_N\}$) then $N \rtimes_\varphi H$ is the direct product $N \times H$.

Semi-direct product 2

Given a group G and two subgroups H, N how to know if G isomorphic to $N \rtimes_\varphi H$ for some φ?

Again three conditions:

- N normal subgroup.
- $N \cap H = \{1\}$.
- $NH = G$.

If the above are satisfied, G isomorphic to $N \rtimes_\varphi H$, where $\varphi(h) =$ conjugation by h of every element in N.

Wreath product

Consider a direct sum $\bigoplus_{x\in H} G$ with index set a group H.

There is a natural action of H on the direct sum:

$$\varphi : H \to \mathrm{Aut}\left(\bigoplus_{h\in H} G\right), \ \varphi(h)f(x) = f(h^{-1}x), \ \forall x \in H .$$

Thus, we define the semidirect product

$$\left(\bigoplus_{h\in H} G\right) \rtimes_{\varphi} H . \tag{1}$$

The semidirect product (1) is called the wreath product of G with H, and it is denoted by $G \wr H$.

The wreath product $G = \mathbb{Z}_2 \wr \mathbb{Z}$ is called the lamplighter group. Its name comes from the way in which $\varphi(1)$ acts.

Finitely generated groups

Given $S \subset G$ and $H \leq G$, H is generated by S (we write $H = \langle S \rangle$) if one of the following equivalent statements is true

- H is the smallest subgroup of G containing S;
- $H = \bigcap_{S \subset K \leq G} K$;
- $H = \{s_1^{\pm 1} s_2^{\pm 1} \dots s_n^{\pm 1} \mid n \in \mathbb{N}, s_i \in S\} \cup \{\mathrm{id}\}$.

- If $H = G$ then S is called generating set.
- If S finite then G is called finitely generated.
- If $S = \{x\}$ then $\langle x \rangle$ cyclic subgroup generated by x.
- Rank of $G =$ minimal number of generators.

Finitely generated groups 2

- If G is finitely generated then G is countable.
- There are uncountably many non-isomorphic finitely generated groups.

- G finitely generated $\Rightarrow G/N$ finitely generated, for any N normal subgroup.
- Not inherited by subgroups (not even normal).
- G, H finitely generated $\Rightarrow G \wr H$ finitely generated (Ex. Sheet 1).

  But $\bigoplus_{x \in H} G$ not finitely generated if H infinite.
- If N, H finitely generated and

$$\{1\} \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow \{1\}, \qquad (2)$$

then G finitely generated (Ex. Sheet 1).

Free groups

What is "the largest (infinite) group" generated by n elements ?

Finite sets: A larger than $B \Leftrightarrow \mathrm{card}(A) \geq \mathrm{card}(B) \Leftrightarrow$ there exists $f : A \to B$ onto.

Infinite groups: We look for a group $G = \langle X \rangle$, $\mathrm{card}(X) = n$, such that for every group $H = \langle Y \rangle$, $\mathrm{card}(Y) = n$, a bijection $X \to Y$ extends to an onto group homomorphism.

Clearly cannot be done for any group G, e.g. if G is abelian then H would have to be abelian.

So G must be a group with no prescribed relation ("free").