Almost everywhere convergence vs. convergence in measure

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Suppose $E \subset \mathbb{R}^n$ be a measurable subset. $f, f_1, f_2, \ldots$ are measurable functions E which are either real-valued or complex-valued.

Lemma 1. *Suppose (f_i) converges to f a.e. in E and $|E| < \infty$. Then (f_i) converges to f in measure.*

The condition $|E| < \infty$ cannot be dropped. A counter-example is: $E = \mathbb{R}$, $f_i = \chi_{[-i,i]}$, $f = 1$.

Proof. Fix $\varepsilon, \delta > 0$. For $i \geq 1$, let

$$Z_i = \{x : \exists j \geq i \text{ such that } |f_j(x) - f(x)| > \delta\} = \bigcup_{j \geq i} \{|f_j - f| > \delta\},$$
$$Z = \bigcup_{i \geq 1} Z_i \subset \{x : \limsup_{j \to \infty} |f_j(x) - f(x)| \geq \delta\}.$$

Then $Z_1 \supseteq Z_2 \supseteq \cdots$, and, since $f_j \to f$ a.e., $|Z| = 0$. Since $|E| < \infty$, $|Z_i| \to |Z| = 0$. In particular, there exists N such that

$$|Z_N| < \varepsilon,$$

which implies (by the definition of Z_N) that

$$|\{|f_j - f| > \delta\}| \leq |Z_N| < \varepsilon \quad \forall\ j \geq N.$$

Since $\varepsilon, \delta > 0$ are arbitrary, this means (f_n) converges to f in measure. $\square$

Lemma 2. *Suppose (f_i) converges to f in measure in E. Then there is a subsequence (f_{i_j}) which converges a.e. to f.*

Proof. By convergence in measure, we can inductively construct $i_j \nearrow \infty$ such that

$$|\{|f_k - f| > 1/j\}| < 2^{-j} \text{ for all } k \geq i_j.$$

Let

$$Z_j = \{|f_{i_j} - f| > 1/j\},$$
$$\bar{Z}_j = \bigcup_{\ell \geq j} Z_j,$$
$$Z = \bigcap_{j \geq 1} \bar{Z}_j.$$

Then

$$|\bar{Z}_j| \leq \sum_{\ell \geq j} |Z_j| \leq 2^{-(j-1)},$$

and so

$$|Z| = 0.$$

Since $|f - f_{i_\ell}| \leq 1/\ell$ in $Z_\ell^c \supseteq \bar{Z}_j^c$ for $\ell \geq j$, this implies that

$$|f - f_{i_\ell}| \leq 1/\ell \text{ in } \bar{Z}_j^c \text{ for all } \ell \geq j.$$

In particular,

$$\lim_{\ell \to \infty} |f - f_{i_\ell}| = 0 \text{ in } \bar{Z}_j^c.$$

Hence

$$\lim_{\ell \to \infty} |f - f_{i_\ell}| = 0 \text{ in } \bigcup_{j \geq 1} \bar{Z}_j^c = Z^c.$$

We are done as $|Z| = 0$. □

LEMMA 3. *Suppose (f_i) is Cauchy in measure. Then there is a subsequence (f_{i_j}) which converges a.e.*

Proof. By Cauchy-ness in measure, we can inductively construct $i_j \nearrow \infty$ such that

$$|\{|f_k - f_\ell| > 2^{-j}\}| < 2^{-j} \text{ for all } k, \ell \geq i_j.$$

Let

$$\begin{aligned} Z_j &= \{|f_{i_{j+1}} - f_{i_j}| > 2^{-j}\}, \\ \bar{Z}_j &= \bigcup_{\ell \geq j} Z_j, \\ Z &= \bigcap_{j \geq 1} \bar{Z}_j. \end{aligned}$$

As in the previous lemma, we have $|\bar{Z}_j| \leq 2^{-(j-1)}$ and $|Z| = 0$. Also, we have

$$|f_{i_{\ell+1}} - f_{i_\ell}| \leq 2^{-\ell} \text{ in } Z_\ell^c \supseteq \bar{Z}_j^c \text{ for all } \ell \geq j.$$

Hence, by telescoping,

$$|f_{i_\ell} - f_{i_m}| \leq \sum_{q=m}^{\ell-1} 2^{-q} \leq 2^{-(j-1)} \text{ in } \bar{Z}_j^c \text{ for all } \ell > m \geq j.$$

Thus the sequence $(f_{i_\ell})_{\ell \geq j}$ converges uniformly on $\bar{Z}_j^c$. It follows that (f_{i_ℓ}) converges on $\cup_j \bar{Z}_j^c = Z^c$. As $|Z| = 0$, we are done. □

Lemma 4. *If (f_i) is Cauchy in measure, then (f_i) converges in measure.*

Proof. Arguing as in the previous proof, we obtain a sequence (f_{i_j}) which converges a.e. to some measurable function f.

Fix $\varepsilon, \delta > 0$. We have

$$\{|f_i - f| > \delta\} \subset \{|f_i - f_{i_j}| \geq \delta/2\} \cup \{|f_{i_j} - f| > \delta/2\} \quad \forall\, i_j.$$

By Cauchy-ness in measure, we can find N such that

$$\{|f_i - f_m| \geq \delta/2\} < \varepsilon/2 \quad \forall i, m \geq N.$$

By a.e. convergence of (f_{i_j}) to f, we can find J such that $i_J \geq N$ and

$$|\{|f_{i_J} - f| > \delta/2\}| < \varepsilon/2.$$

It then follows that

$$|\{|f_i - f| > \delta\}|| < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall\, i \geq N.$$

This gives the convergence in measure of (f_i) to f. □