Infinite Groups

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First, two announcements

The lecture from Thursday October 31st is rescheduled to Tuesday November 5th, in L1.

For Exercise Sheets 1 and 2, please consider the entire set as Section B: they are all core questions.

About Mathematics

More than any other science, Mathematics is not limited by our physical bounds.

It allows us to push forever outward in our exploration, taking the measure of objects and phenomena far removed from our grasp, independently of the technical advancement of the time.

Using geometry, a well and a sundial, Erathostenes could measure the circumference of the Earth with an error of 50 miles, around 230 BC.

Cayley graph a tree

Proposition

The group G is a free group generated by Y if and only if $\mathrm{Cayley}(G;S)$ is a tree, where $S = Y \sqcup Y^{-1}$.

Proof of $\Rightarrow$. Assume $\mathrm{Cayley}(G;S)$ contains a circuit.
We choose an orientation on the circuit and read the label on the thus oriented circuit: it is a word $s_1 \dots s_k$ equal to 1 in G.
The word is then not reduced: for some i we have $s_{i+1} = s_i^{-1}$.
Thus the oriented circuit contains two consecutive edges $e_i = (x_i, y_i)$ and $e_{i+1} = (y_i, x_i)$, contradiction.
Proof of $\Leftarrow$. $\mathrm{Cayley}(G;S)$ connected $\Rightarrow$ $G = \langle Y \rangle$.
By Universal Property there exists $\varphi : F(Y) \to G$ onto homomorphism.
Assume there exists $s_1 \dots s_k$ reduced word in S, equal to 1 in G.
Assume k is minimal. A path labeled by $s_1 \dots s_k$ in $\mathrm{Cayley}_{\mathrm{dir}}(G;Y)$ is a loop without spikes. Minimality of k implies it is simple. $\square$

Group acting freely on a tree

Corollary

A free group acts freely on a tree.

free action = the stabilizer of every point is $\{1\}$.

The converse of the above is also true, that is

Theorem

A group is free if and only if it acts freely on a simplicial tree.

The proof of this is part of the "Geometric Group Theory" course.

Corollary

Every subgroup of a free group is free.

Rank of a free group

We mention a few other results, without proof.

Proposition

$F(X)$ is isomorphic to $F(Y)$ if and only if $\mathrm{card}(X) = \mathrm{card}(Y)$.

Notation

We denote by F_n the group $F(X)$ with $\mathrm{card}(X) = n$*, unique up to isomorphism by the above.*

Proposition

The rank of $F(X)$ is $\mathrm{card}(X)$.

NB $F(X) \leqslant F(Y)$ does not imply $\mathrm{card}(X) \leqslant \mathrm{card}(Y)$.

Exercise

Every free group of countable rank can be embedded as a subgroup of F_2 (Exercise 7, Sheet 1).

Presentations of groups

How to fully describe a group?

- Table of multiplication if G is finite;
- Free groups.

Answer in general case: by generators and relations.

Example

$\mathbb{Z}^2$ is the group generated by two elements a, b satisfying the relation

$$ab = ba \Leftrightarrow [a, b] = 1.$$

We write $\mathbb{Z}^2 = \langle a, b \mid [a, b] = 1 \rangle$ or simply $\mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$.

Presentations of groups 2

In general let $G = \langle S \rangle$. By Universal property, $\exists$ an onto homomorphism

$$\pi_S : F(S) \to G$$

whence G isomorphic to $F(S)/\ker(\pi_S)$.
The elements of $\ker(\pi_S)$ are called relators or relations for G and the generating set S.
We are interested in minimal subsets R of $\ker(\pi_S)$ such that $\ker(\pi_S)$ is normally generated by R.
$N \triangleleft G$ is normally generated by $R \subset N$ (or N normal closure of R) if one of the following equivalent properties is satisfied:

- N is the smallest normal subgroup of G containing R;
- $N = \bigcap_{R \subset K \triangleleft G} K$;
- $N = \{r_1^{x_1} \cdots r_n^{x_n} \mid n \in \mathbb{N}, r_i \in R \cup R^{-1}, x_i \in G\} \cup \{1\}$.

Notation

$a^b = bab^{-1}$, $A^B = \{a^b \mid a \in A, b \in B\}$. $N = \langle\langle R \rangle\rangle$.

Presentation of groups 3

Let $R \subset \ker(\pi_S)$ be such that $\ker(\pi_S) = \langle\langle R \rangle\rangle$.
We say that the elements $r \in R$ are defining relators.
The pair (S, R) defines a presentation of G.
We write $G = \langle S \mid r = 1, \forall r \in R \rangle$ or simply $G = \langle S \mid R \rangle$.
Formally, it means G is isomorphic to $F(S)/\langle\langle R \rangle\rangle$.
Equivalently:

- $\forall g \in G$, $g = s_1 \cdots s_n$, for some $n \in \mathbb{N}$ and $s \in S \cup S^{-1}$;
- $w \in F(S)$ satisfies $w =_G 1$ if and only if in $F(S)$

$$w = \prod_{i=1}^{m} r_i^{x_i}, \text{ for some } m \in \mathbb{N}, r_i \in R, x_i \in F(S).$$

Examples of group presentations

1. $\langle a_1, \dots, a_n \mid [a_i, a_j], 1 \leqslant i, j \leqslant n \rangle$ is a finite presentation of $\mathbb{Z}^n$;
2. $\langle x, y \mid x^n, y^2, yxyx \rangle$ is a presentation of the finite dihedral group D_{2n} ;
3. $\langle x, y \mid x^3, y^2, [y, x] \rangle$ is a presentation of the cyclic group $\mathbb{Z}_6$;
4. $\langle x_1, \dots, x_n \mid x_i^2, (x_i x_j)^{m_{ij}} \rangle$, is a presentation of a Coxeter group;
5. the Integer Heisenberg group:

$$H_{2n+1}(\mathbb{Z}) := \langle x_1, \dots, x_n, y_1, \dots, y_n, z\, ;$$

$$[x_i, z] = 1, [y_j, z] = 1, [x_i, x_j] = 1, [y_i, y_j] = 1, [x_i, y_j] = z^{\delta_{ij}}, 1 \leqslant i, j \leqslant n \rangle.$$

The Integer Heisenberg group:

$$H_{2n+1}(\mathbb{Z}) := \langle x_1, \dots, x_n, y_1, \dots, y_n, z\,;$$

$$[x_i, z] = 1, [y_j, z] = 1, [x_i, x_j] = 1, [y_i, y_j] = 1, [x_i, y_j] = z^{\delta_{ij}}, 1 \leqslant i, j \leqslant n\rangle.$$

$$H_{2n+1}(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a_1 & a_2 & \dots & \dots & a_n & c \\ 0 & 1 & 0 & \dots & \dots & 0 & b_n \\ 0 & 0 & 1 & \dots & \dots & 0 & b_{n-1} \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 & b_2 \\ 0 & 0 & \dots & \dots & 0 & 1 & b_1 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} ;\ a_i, b_j, c \in \mathbb{Z} \right\}$$

Finitely presented groups

A group G is called finitely presented if it admits a finite presentation.

NB While there are continuously many finitely generated groups, there are only countably many finitely presented groups.

Sometimes it is difficult, and even algorithmically impossible, to find a finite presentation of a finitely presented group.

Proposition (Universal Property Generalized)

Let $G = \langle S | R \rangle$. A map $\psi : S \to H$ with target a group H can be extended (uniquely) to a group homomorphism $\Phi : G \to H$ iff for every $r \in R$, $r = s_1 \ldots s_n$, $\psi(s_1) \ldots \psi(s_n) = 1$ in H.

Generalization of the Universal Property

Proposition

Let $G = \langle S|R\rangle$. A map $\psi : S \to H$ with target a group H can be extended (uniquely) to a group homomorphism $\Phi : G \to H$ iff for every $r \in R$, $r = s_1 \dots s_n$, $\psi(s_1)\dots\psi(s_n) = 1$ in H.

Proof. The Universal property of free groups $\Rightarrow \psi$ extends to $\tilde{\psi} : F(S) \to H$.

$\langle\langle R\rangle\rangle = \langle R^{F(S)}\rangle$ is generated by elements of the form grg^{-1}, where $g \in F(S), r \in R$.
$\tilde{\psi}(grg^{-1}) = 1 \Rightarrow \langle\langle R\rangle\rangle \leqslant \ker(\tilde{\psi}) \Rightarrow \tilde{\psi}$ defines $\Phi : F(S)/\langle\langle R\rangle\rangle \to H$.

Uniqueness: because every homomorphism is entirely determined by its restriction to a generating set. $\square$