Problem Sheet 1: Solutions

1. Taking the suggested dot product and using the summation convention we have that

$$
\begin{aligned}
k_i \int_C (\mathbf{f} \wedge \mathbf{t})_i \, \mathrm{d}s = \int_C k_i (\mathbf{f} \wedge \mathbf{t})_i \, \mathrm{d}s &= -\int_C (\mathbf{f} \wedge \mathbf{k})_i t_i && \text{[Scalar triple product]} \\
&= -\int_S n_i \big[\nabla \wedge (\mathbf{f} \wedge \mathbf{k})\big]_i && \text{[Stokes' Theorem]} \\
&= -\int_S n_i \varepsilon_{ijk} \partial_j [\varepsilon_{klm} f_l k_m] \, \mathrm{d}S \\
&= -\int_S n_i k_m (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_j f_l \, \mathrm{d}S && \text{[Using the hint and } \mathbf{k} \text{ constant]} \\
&= -\int_S n_i (k_j \partial_j f_i - k_i \partial_j f_j) \, \mathrm{d}S \\
&= k_i \int_S n_i \partial_j f_j \, \mathrm{d}S - k_i \int_S n_j \partial_i f_j \, \mathrm{d}S && \text{[letting } i \leftrightarrow j\text{].}
\end{aligned}
$$

Since $\mathbf{k}$ is an arbitrary vector we must, in fact, have the desired result.

Now, letting $\mathbf{f} = \gamma \mathbf{n}$ we have

$$
\begin{aligned}
-\left[\gamma \int_C \boldsymbol{\nu} \, \mathrm{d}s\right]_i = \left[\int_C \gamma (\mathbf{n} \wedge \mathbf{t}) \, \mathrm{d}s\right]_i &= \left[\int_C \mathbf{f} \wedge \mathbf{t} \, \mathrm{d}S\right]_i \\
&= \int_S [n_i \nabla \cdot (\gamma \mathbf{n}) - n_j \partial_i (\gamma n_j)] \, \mathrm{d}S && \text{[Using the earlier result]} \\
&= \int_S \{n_i [\gamma \partial_j n_j + (n_j \partial_j \gamma = 0)] && \text{[Since } \gamma \text{ is constant normal to } S\text{]} \\
&- n_j (n_j \partial_i \gamma + \gamma \partial_i n_j)\} \, \mathrm{d}S \\
&= \int_S [n_i \gamma (\nabla \cdot \mathbf{n}) - \partial_i \gamma - \gamma \partial_i (n_j n_j / 2)] \, \mathrm{d}S \\
&= \int_S [n_i \gamma (\nabla \cdot \mathbf{n}) - \partial_i \gamma] \, \mathrm{d}S && \text{[Since } n_j n_j = \mathbf{n} \cdot \mathbf{n} = 1\text{]} \\
&= \int_S [\mathbf{n} \gamma (\nabla \cdot \mathbf{n}) - \nabla \gamma]_i \, \mathrm{d}S,
\end{aligned}
$$

as required.

2. We are given that

$$\ell_c^2 \frac{h_{xx}}{(1+h_x^2)^{3/2}} = h,$$

which we multiply by h_x and integrate once to obtain

$$A\ell_c^2 - \ell_c^2(1+h_x^2)^{-1/2} = \tfrac{1}{2}h^2,$$

for some constant of integration A.

Now, as $x \to +\infty$, $h \to 0$, $h_x \to 0$ so that $A = 1$ and

$$\tfrac{1}{2}h^2 = \ell_c^2 \left[1 - (1+h_x^2)^{-1/2}\right]. \tag{1}$$

At $x = 0$, $h_x = -\cot\theta$ so that

$$\tfrac{1}{2}h_0^2 = \ell_c^2 \left[1 - (1+\cot^2\theta)^{-1/2}\right] = \ell_c^2(1-\sin\theta),$$

i.e.

$$h_0 = \pm\ell_c \left[2(1-\sin\theta)\right]^{1/2},$$

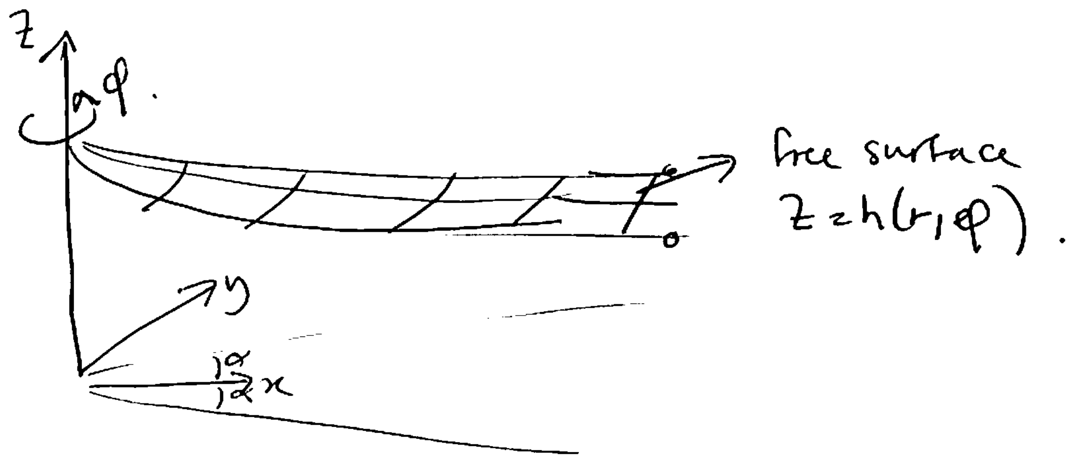
as desired. Based on simple geometry, we expect to take the positive square root if $\theta < \pi/2$ and the negative root if $\theta > \pi/2$, though of course this depends on the chosen sign convention for h!

The area of displaced liquid is given by

$$\begin{aligned} A &= \int_0^\infty h \, \mathrm{d}x = \int_{-\cot\theta}^0 \frac{\ell_c^2}{(1+h_x^2)^{3/2}} \, \mathrm{d}(h_x) && \text{[Using the Laplace–Young equation]} \\ &= \ell_c^2 \int_{-\theta}^{-\pi/2} \frac{-1/\sin^2 u}{1/|\sin u|^3} \, \mathrm{d}u && \text{[Letting } h_x = \cot u\text{]} \\ &= \ell_c^2 \cos\theta. \end{aligned}$$

The weight of liquid displaced is $\rho g A = \gamma\cos\theta$. The vertical force provided by surface tension acting at the contact line is also $\gamma\cos\theta$. This result is thus a special case of the generalized Archimedes' principle discussed in lectures.

3. A schematic sketch is shown below.



We have the general, coordinate independent, form of the Laplace–Young equation

$$\rho g h = \gamma\kappa = -\gamma\nabla\cdot\mathbf{n}.$$

The equation of the free surface is $0 = z - h(r,\phi)$ and so the unit normal vector is

$$\mathbf{n} = \frac{(-h_r, -h_\phi/r, 1)}{\left(1 + h_r^2 + h_\phi^2/r^2\right)^{1/2}}.$$

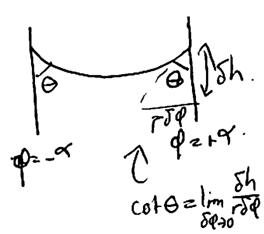
For small deflections (linearising) we have $\mathbf{n} \approx (-h_r, -h_\phi/r, 1)$ so that

$$\nabla\cdot\mathbf{n} = -\frac{1}{r}\frac{\partial}{\partial r}(rh_r) - \frac{1}{r^2}\frac{\partial^2 h}{\partial\phi^2} = -\nabla^2 h$$

and we immediately have that the Laplace–Young equation takes the form

$$h = \ell_c^2\nabla^2 h$$

where $\ell_c = (\gamma/\rho g)^{1/2}$ is the capillary length, as usual.



To obtain the correct contact angle condition we take a cross-section through the interface, at constant r, say and look along the $-\mathbf{i}$ direction (back towards the origin). We see immediately from the figure above that

$$\left.\frac{1}{r}\frac{\partial h}{\partial\phi}\right|_{\phi=\pm\alpha} = \pm\cot\theta.$$

In our derivation, we have assumed that the the meniscus slope is small, in particular, $|\nabla h|^2 \ll 1$. Given the above boundary condition and this requirement, we also need to ensure that $\cot\theta \ll 1$ and hence $|\theta - \frac{\pi}{2}| \ll 1$, as required.

Uniqueness We will first show that, if we can find a solution, it must be unique.

As usual, we proceed by contradiction assuming that there are two distinct solutions of the problem, $h_1 \neq h_2$ both satisfying the Laplace–Young equation and the relevant boundary conditions (at $\theta = \pm\alpha$ or at $\theta = \pm\pi/4$ — it doesn't matter which). Letting $w = h_2 - h_1$ it is obvious that

$$\nabla^2 w = w/\ell_c^2 \tag{2}$$

and

$$\frac{1}{r}\frac{\partial w}{\partial \phi}\bigg|_{\phi=\pm\alpha} = 0 \tag{3}$$

and, finally, that $w \to 0$ far from the walls (i.e. as $r \to \infty$ with $\phi \neq \pm\alpha$).

Letting S be the projection of the interface onto the (x, y) plane, which is bounded by the curve C, we consider the integral

$$\begin{aligned}
\int_S w\nabla^2 w \, \mathrm{d}S &= \int_S w^2/\ell_c^2 \geq 0 && \text{[Using (2)]} \\
&= \int_S \left[\nabla\cdot(w\nabla w) - (\nabla w)^2\right] \, \mathrm{d}S \\
&= \int_C (\mathbf{n}\cdot\nabla w = 0) \, \mathrm{d}s - \int_S (\nabla w)^2 \, \mathrm{d}S && \text{[Using the boundary condition (3)]} \\
&= -\int_S (\nabla w)^2 \, \mathrm{d}S \leq 0, && (4)
\end{aligned}$$

which is the contradiction we sought. Hence the solution must be unique.

Finding a solution To find a solution, it is enough to check that the solution given satisfies the Laplace–Young equation and the boundary conditions.

Another approach is to introduce rotated coordinates (X, Y) so that the 90° wedge coincides with the X and Y axes, i.e. we let

$$X = \frac{x-y}{\sqrt{2}}, \quad Y = \frac{x+y}{\sqrt{2}}.$$

Then the Laplace–Young equation for the interface shape $H(X, Y)$ becomes

$$H_{XX} + H_{YY} = H/\ell_c^2$$

with boundary conditions

$$H_X(X=0) = -\cot\theta, \quad H_Y(Y=0) = -\cot\theta. \tag{5}$$

and decay conditions far away from the wall.

Searching for separable solutions of the form $H(X,Y) = \xi(X)\eta(Y)$ we find that

$$\frac{\xi''}{\xi} + \frac{\eta''}{\eta} = 1/\ell_c^2$$

which gives solutions of the form

$$H = A\exp(-\alpha X/\ell_c)\exp(-\beta Y/\ell_c)$$

where $1 = \alpha^2 + \beta^2$. Applying the boundary conditions (5) we find that we must combine two solutions of this form: one with $\alpha = 1, \beta = 0$ and the other with $\alpha = 0, \beta = 1$. We therefore have

$$H = \ell_c \cot\theta \left\{ e^{-X/\ell_c} + e^{-Y/\ell_c} \right\}.$$

Noting that $X = (x-y)/\sqrt{2} = r\sin(\pi/4 - \phi)$ and that $Y = (x+y)/\sqrt{2} = r\sin(\phi + \pi/4)$ we therefore have that

$$h(r,\phi) = \ell_c \cot\theta \left\{ e^{-r\sin(\pi/4-\phi)/\ell_c} + e^{-r\sin(\phi+\pi/4)/\ell_c} \right\},$$

as required.

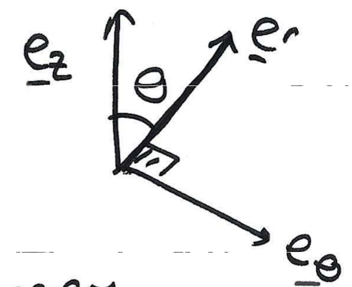
Sheet 1, Q4

(i) By symmetry $\underline{F} = F_3 \underline{e_z}$.

ii) For r^2 term, $\psi = -\frac{Ur^2\sin^2\theta}{2}$

$$u_r = \frac{1}{r^2\sin\theta}\psi_\theta = -U\cos\theta, \quad u_\theta = -\frac{1}{r\sin\theta}\psi_r = U\sin\theta$$

$$\therefore \underline{u} = -U\cos\theta\,\underline{e_r} + U\sin\theta\,\underline{e_\theta}$$
$$= -U\underline{e_z}.$$



$\underline{u}$ constant

$$\therefore 0 = -\nabla p + \rho\underline{g}$$

and

$$F_i\Big|_{r^2 \text{ term}} = -\int_{\text{Bubble}} p n_i \, dS \overset{\text{Divergence theorem}}{=} \int_{\text{Bubble}} -(\nabla p)_i \, dV$$
$$= -\rho(\underline{g})_i\left(\frac{4\pi a^3}{3}\right) = \rho g\left(\frac{4\pi a^3}{3}\right)(\underline{e_z})_i$$

$$\therefore F_3\Big|_{r^2 \text{ term}} = \frac{4\pi a^3}{3}\rho g.$$

iii)

$$(F_3)\Big|_{1/r \text{ term}} = \int_{\text{Bubble}} (T_{3j}) n_j \, dS = \int_{\substack{\text{Sphere,} \\ \text{radius } R,\ R\to\infty}} (T_{3j}) n_j \, dS$$

(Divergence theorem, using $\partial_j(T_{ij}) = 0$, by Stokes Equation.)

$$(T_{3j})\Big|_{1/r \text{ term}} \sim 1/r^4 \quad \text{e.g. } u_r \sim 1/r^3 \text{ and } \frac{\partial u_r}{\partial r} \sim T \sim 1/r^4$$

$$\therefore F_3\Big|_{1/r \text{ term}} \sim \int_{\substack{\text{Sphere, radius} \\ R\to\infty}} \frac{1}{R^4} R^2 \sin\theta \, d\theta \, d\varphi \sim \frac{1}{R^2} \to 0 \text{ as } R \to \infty$$

$\therefore$ No force generated.

(iv) Bubble moving at Speed U.

The alpha term generates no force by part (iii) .. therefore can ignore

$\underline{F}\big|_{r\ \text{term}}$ = Force due to streamfunction $\underbrace{C}_{\text{coefficient}} r\sin^2\theta$ } Same to within coefficients.

Coefficient $3Va/4$

Force due to stream function $\dfrac{3Var\sin^2\theta}{4}$ is $-6\pi\mu Va\,\underline{e}_z$

$$\therefore\ \underline{F}\big|_{r\ \text{term}} = C\ \frac{(-6\pi\mu Ua)\,\underline{e}_z}{3Ua/4}$$

← Divide by $3Ua/4$ and multiply by C as Stokes Equations linear

$$= \left[\frac{Ua}{2} + \frac{1}{6}\frac{\gamma' a^2}{\mu}\right]\left[-8\pi\mu\right]$$

$$= -4\pi\mu Ua - \tfrac{4}{3}\gamma' a^2\pi$$

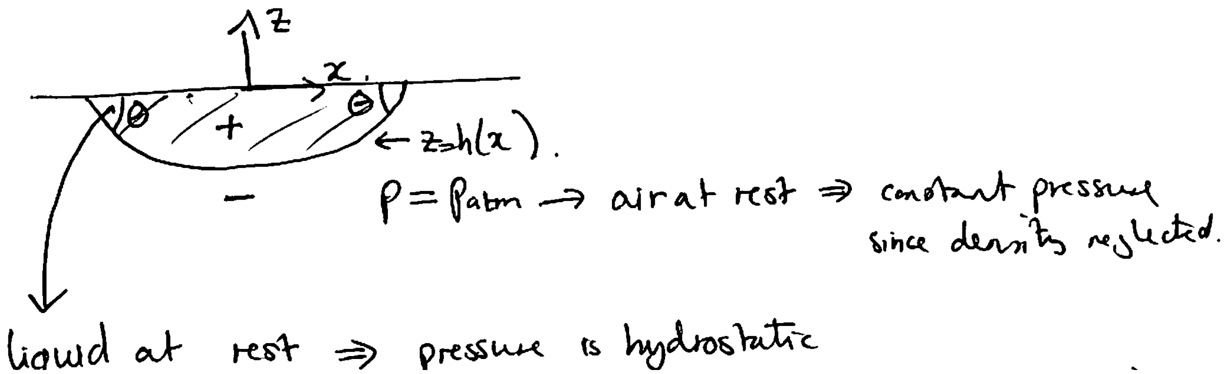
Bubble stationary. $U = 0$. Force $= 0$.

$$\therefore\ 0 = \text{Force due to } r^2 \text{ term} + \underbrace{\text{Force due to } 1/r \text{ term}}_{0} + \text{Force due to } r \text{ term}$$

$$= \frac{4\pi a^3 \rho g}{3} - \tfrac{4}{3}\gamma' a^2\pi$$

$$\therefore\ \boxed{\gamma' = \rho g a}$$

5 The scenario is as shown in the figure below.



Since the liquid is static, the pressure within it is hydrostatic, i.e.

$$p = p_0 - \rho g z$$

where the z coordinate is measured vertically upwards and p_0 is some reference pressure ($p_0 \neq p_{\text{atm}}$).

Because of surface tension, there is a pressure jump across the interface:

$$\begin{aligned}(p_+ - p_-)|_{z=h(x)} &= \gamma\kappa \approx \gamma h_{xx} \\ &= p_0 - \rho g h(x) - p_{\text{atm}} \qquad \text{[Using hydrostatic pressure in the liquid]}\end{aligned}$$

from which we immediately have

$$h + \ell_c^2 h_{xx} = \frac{p_0 - p_{\text{atm}}}{\rho g} \tag{6}$$

with $\ell_c^2 = \gamma/\rho g$ as usual. Note that (6) is slightly different from the usual Laplace–Young equation in that it has a source term on the RHS *and* the solutions of the homogeneous problems are oscillatory rather than the usual exponential decay.

Differentiating (6) with respect to x we obtain the required third order ODE; solving either this ODE or (6) we have solutions of the form

$$h(x) = A + B \sin x/\ell_c + C \cos x/\ell_c.$$

The coefficients A, B, C are to be determined from the boundary conditions

$$\begin{aligned} h(\pm x_0) &= 0 \\ h_x(\pm x_0) &= \pm \tan\theta \approx \pm\theta \qquad \text{[Since } \theta \ll 1\text{]}\end{aligned}$$

From the first boundary condition, we have

$$A - B \sin x_0/\ell_c + C \cos x_0/\ell_c = A + B \sin x_0/\ell_c + C \cos x_0/\ell_c = 0$$

from which either $B = 0$ or $x_0/\ell_c = n\pi$.

From the second boundary condition, we have

$$\pm\ell_c\theta = B\cos x_0/\ell_c \mp C\sin x_0/\ell_c$$

from which either $B = 0$ or $x_0/\ell_c = (n + 1/2)\pi$.

For consistency between the two sets of boundary conditions, we must have $B = 0$ (i.e. the drop is symmetric) and we immediately find that

$$h(x) = \theta\ell_c\left[\cot x_0/\ell_c - \frac{\cos x/\ell_c}{\sin x_0/\ell_c}\right]$$

For this solution, $|h_x| = \theta|\sin x/\ell_c|/\sin x_0/\ell_c \leq \theta/\sin x_0/\ell_c$ so the small slope approximation is valid provided that $\theta \ll \sin x_0/\ell_c$.

The area of the drop is

$$A = \int_{-x_0}^{x_0} -h\ \mathrm{d}x = 2\theta\ell_c^2\left[1 - \frac{x_0}{\ell_c}\cot x_0/\ell_c\right].$$

As $x_0/\ell_c \to \pi$, $A \to \infty$. This suggests that infinitely large droplets can be supported beneath a horizontal plate. Intuitively, we expect that droplets should fall off the plate if they become too large. The problem with our linearised analysis is that as $x_0/\ell_c \to \pi$ there are no values of θ for which our linearised analysis is self-consistent — the small-slope approximation breaks down in this limit.