Infinite Groups

Cornelia Druțu

University of Oxford

Part C course MT 2024, Oxford

This week's quotation, and some perspective

William Thurston: "Experience has shown repeatedly that a mathematical theory with a rich internal structure generally turns out to have significant implications for the understanding of the real world, often in ways no one could have envisioned before the theory was developed."

Speaking of Thurston, from this week on we begin to study groups closely connected to two of his eight geometries.

Thurston conjectured that every 3-dimensional manifold has a standard decomposition into pieces that can be endowed with one of these eight geometries. This is called the Geometrisation conjecture. It was proved by Grigori Perelman, whose work was recognized with a Fields Medal in 2006.

Finite presentability is independent of the generating set

A group G is called finitely presented if it admits a finite presentation.

Proposition

Assume $G = \langle S \mid R \rangle$ finite presentation, and $G = \langle X \mid T \rangle$ is such that X is finite. Then $\exists$ finite subset $T_0 \subset T$ such that $G = \langle X \mid T_0 \rangle$.

Proof. $\forall s \in S$ $\exists a_s(X)$ word in X s.t. $s = a_s(X)$ in G. (involves a choice)
The map $i_{SX} : S \to F(X)$, $i_{SX}(s) = a_s(X)$ extends to a unique homomorphism $p : F(S) \to F(X)$ (rewriting homomorphism).
We have that $\pi_S = \pi_X \circ p$.
Likewise, $\forall x \in X$ $\exists b_x(S)$ in S s.t. $x = b_x(S)$.
The map $i_{XS} : X \to F(S)$, $i_{XS}(x) = b_x(S)$, extends to homomorphism $q : F(X) \to F(S)$ (another rewriting homomorphism).
As previously $\pi_S \circ q = \pi_X$.

Finite presentability is independent of the generating set 2

For every $x \in X$,

$$\pi_X(x) = \pi_S(q(x)) = \pi_X(p(q(x))).$$

Whence for every $x \in X$, $x^{-1}p(q(x)) \in \ker(\pi_X)$.
Let N be the normal subgroup of $F(X)$ normally generated by

$$\{p(r) \mid r \in R\} \cup \{x^{-1}p(q(x)) \mid x \in X\}.$$

We have that $N \leqslant \ker(\pi_X)$. Goal: prove equality.
There is a natural projection

$$Q : F(X)/N \to F(X)/\ker(\pi_X).$$

Finite presentability is independent of the generating set 3

Let $\bar{p} : F(S) \to F(X)/N$ be the homomorphism induced by p.
$\bar{p}(r) = 1$ for all $r \in R \Rightarrow \bar{p}$ induces a homomorphism

$$Q' : F(S)/\ker(\pi_S) \to F(X)/N.$$

Note that the domain of Q' is isomorphic to G, and Q' is onto: $F(X)/N$ is generated by $xN = p(q(x))N$, and the latter is the image under Q' of $q(x)\ker(\pi_S)$.
Consider the homomorphism

$$Q \circ Q' : F(S)/\ker(\pi_S) \to F(X)/\ker(\pi_X)$$

.
The isomorphism $G \to F(S)/\ker(\pi_S)$ sends every $x \in X$ to $q(x)\ker(\pi_S)$.
The isomorphism $G \to F(X)/\ker(\pi_X)$ sends every $x \in X$ to $x\ker(\pi_X)$.
Note that $Q \circ Q'\,(q(x)\ker(\pi_S)) = Q(xN) = x\ker(\pi_X)$, whence $Q \circ Q'$ isomorphism $\Rightarrow Q'$ injective $\Rightarrow Q'$ isomorphism $\Rightarrow Q$ isomorphism $\Rightarrow N = \ker(\pi_X)$.

Finite presentability is independent of the generating set 4

In particular, $\ker(\pi_X)$ is normally generated by the finite set of relators

$$\mathfrak{R} = \{p(r) \mid r \in R\} \cup \{x^{-1}p(q(x)) \mid x \in X\}.$$

Since $\mathfrak{R} \subset \langle\langle T \rangle\rangle$, every relator $\rho \in \mathfrak{R}$ can be written as a product

$$\prod_{i \in I_\rho} t_i^{v_i}$$

with $v_i \in F(X), t_i \in T$ and I_ρ finite.
Whence $\ker(\pi_X)$ is normally generated by the finite subset

$$T_0 = \bigcup_{\rho \in \mathfrak{R}} \{t_i \mid i \in I_\rho\}$$

of T. □

Finite presentability is independent of the generating set

Proposition

Assume $G = \langle S \mid R \rangle$ finite presentation, and $G = \langle X \mid T \rangle$ is such that X is finite. Then $\exists$ finite subset $T_0 \subset T$ such that $G = \langle X \mid T_0 \rangle$.

Reformulation of the Proposition:

Given a short exact sequence

$$1 \to N \to F(X) \to G \to 1$$

with X finite and G finitely presented, N is normally generated by finitely many elements $n_1, \ldots, n_k$.

This reformulation has a generalization to arbitrary short exact sequences that will appear later on.

Commutators

Notation

$[x, y] = xyx^{-1}y^{-1}$= *the commutator of the elements* x, y *in a group* G

For subsets A, B *in a group* G, $[A, B]$= *the subgroup of* G *generated by all the commutators* $[a, b], a \in A, b \in B$.

For every $x_1, \dots, x_n$ *in a group* G *we denote by* $[x_1, \dots, x_n]$ *the n-fold left-commutator*

$$[[[x_1, x_2], \dots, x_{n-1}], x_n].$$

For subsets $A_1, \dots A_n$ *in a group* G, $[A_1, \dots, A_n]$= *the subgroup of* G *generated by all the commutators* $[a_1, \dots, a_n], a_i \in A_i$.

Nilpotent Groups: first definition

There are two ways of defining nilpotent groups and of measuring "how far they are from being abelian".

First definition: from the group downwards.
The lower central series of a group G,

$$C^1 G \trianglerighteq C^2 G \trianglerighteq \dots \trianglerighteq C^n G \trianglerighteq \dots,$$

is defined inductively by:

$$C^1 G = G\,,\ C^{n+1} G = [C^n G, G]\,.$$

Each $C^k G$ is a characteristic subgroup of G (i.e. for every automorphism $\varphi : G \to G$, $\varphi(C^k G) = C^k G$).
$C^2 G = [G, G] = G'$ is the commutator subgroup, or the derived subgroup, of G.

Nilpotent Groups 2

Definition

G is k-step nilpotent if $C^{k+1}G = \{1\}$. The minimal k for which G is k-step nilpotent is called the (nilpotency) class of G.

Examples

1. Every non-trivial abelian group is nilpotent of class 1.
2. The group $\mathcal{U}_n(\mathbb{K})$ of upper triangular $n \times n$ matrices with 1 on the diagonal and entries in a ring $\mathbb{K}$, is nilpotent of class $n-1$ (see Exercise Sheet 2).
3. The integer Heisenberg group $H_{2n+1}(\mathbb{Z})$ is nilpotent of class 2.

Basic properties

Lemma

If $G = \langle S \rangle$ (S not necessarily finite, G not necessarily nilpotent), then $\forall k$ the subgroup $C^k G$ is generated by the k-fold left commutators in S, together with $C^{k+1} G$.

Proof Induction on k and two formulas:

- $[x, yz] = [x, y]\,[y, [x, z]]\,[x, z]\,;$
- $[xy, z] = [x, [y, z]]\,[y, z]\,[x, z] = [y, z]^x [x, z]\,.$

Corollary

If $G = \langle S \rangle$ is nilpotent, then $C^n G$ is generated by all the k-fold left commutators in S, where $k \geqslant n$. In particular, if G is finitely generated, each subgroup $C^n G$ is finitely generated.