Infinite Groups

Cornelia Druţu

University of Oxford

Part C course MT 2024, Oxford

Nilpotent Groups: first definition

Notation

$[x, y] = xyx^{-1}y^{-1}$ = *the commutator of the elements x, y in a group G.*

For subsets A, B in G, $[A, B] = \langle \{[a, b]; a \in A, b \in B\} \rangle$.

$[x_1, \dots, x_n] = [[[x_1, x_2], \dots, x_{n-1}], x_n]$ *is called an n-fold left commutator.*

For subsets $A_1, \dots A_n$ in a group G,
$[A_1, \dots, A_n] = \langle \{[a_1, \dots, a_n]; a_i \in A_i\} \rangle$.

First definition of nilpotent groups: from the group downwards.
The lower central series of a group G, $C^1G \trianglerighteq C^2G \trianglerighteq \dots \trianglerighteq C^nG \trianglerighteq \dots$, is defined inductively by: $C^1G = G$, $C^{n+1}G = [C^nG, G]$.

G is k-step nilpotent if $C^{k+1}G = \{1\}$. The minimal k = the nilpotency class of G.

Basic properties

Lemma

If $G = \langle S \rangle$ (S not necessarily finite, G not necessarily nilpotent), then $\forall k$ the subgroup $C^k G$ is generated by the k-fold left commutators in S, together with $C^{k+1} G$.

Corollary

If $G = \langle S \rangle$ is nilpotent, then $C^n G$ is generated by all the k-fold left commutators in S, where $k \geqslant n$. In particular, if G is finitely generated, each subgroup $C^n G$ is finitely generated.

Nilpotent Groups: second definition

Second definition: from $\{1\}$ upwards.
The center $Z(H)$ of a group H is composed of all $z \in H$ s.t. $zh = hz, \forall h \in H$.

Given a group G, define inductively an increasing sequence of normal subgroups $Z_i(G) \lhd G$ by:

- $Z_0(G) = \{1\}$.
- If $Z_i(G) \lhd G$ is defined and $\pi_i : G \to G/Z_i(G)$ is the quotient map, then

$$Z_{i+1}(G) = \pi_i^{-1}\left(Z(G/Z_i(G))\right).$$

Note that $Z_{i+1}(G)$ is normal in G, as the inverse image of a normal subgroup of a quotient of G.
In particular,

$$Z_{i+1}(G)/Z_i(G) \cong Z(G/Z_i(G)).$$

Nilpotent Groups: second definition 2

Proposition

G is k-step nilpotent if and only if $Z_k(G) = G$.

Proof Assume that G is nilpotent of class k.
We prove by induction on $i \geqslant 0$ that $C^{k+1-i}G \leqslant Z_i(G)$.
For $i = 0$ we have equality.
Assume that

$$C^{k+1-i}G \leqslant Z_i(G).$$

For every $g \in C^{k-i}G$ and every $x \in G$, $[g, x] \in C^{k+1-i}G \leqslant Z_i(G)$, whence $gZ_i(G)$ is in the center of $G/Z_i(G)$, i.e. $g \in Z_{i+1}(G)$.

For $i = k$ the inclusion becomes $C^1G = G \leqslant Z_k(G)$, hence $Z_k(G) = G$.

Nilpotent Groups: second definition 3

Conversely, assume $Z_k(G) = G$.

We prove by induction on $j \geqslant 1$ that $C^j G \leqslant Z_{k+1-j}(G)$.

For $j = 1$ the two are equal.

Assume the inclusion true for j.

$C^{j+1}G$ generated by $[c, g]$ with $c \in C^j G$ and $g \in G$.

Since $c \in C^j G \leqslant Z_{k+1-j}(G)$, by the definition of $Z_{k+1-j}(G)$, the element c commutes with g modulo $Z_{k-j}(G)$, equivalently $[c, g] \in Z_{k-j}(G)$.

This implies that $[c, g] \in Z_{k-j}(G)$. It follows that $C^{j+1}G \leqslant Z_{k-j}(G)$.

For $j = k + 1$ this gives $C^{k+1}G \leqslant Z_0(G) = \{1\}$, hence G is k-step nilpotent. □

Nilpotent Groups: second definition 4

Definition

The ascending series

$$Z_0(G) = \{1\} \lhd Z_1(G) \lhd \ldots \lhd Z_i(G) \lhd Z_{i+1}(G) \lhd \ldots$$

of normal subgroups of G is called the upper central series of G.

A group G is nilpotent if and only if there exists i such that $Z_i(G) = G$, and its nilpotency class is the minimal k such that $Z_k(G) = G$.

The following example shows that the difference between lower and upper central series of groups can be quite substantial:

Example

We start with the integer Heisenberg group H; it is 2-step nilpotent, $C^2H = H' = Z(H) \cong \mathbb{Z}$.
Take $G = H \times \mathbb{Z}$, 2-step nilpotent. $C^2G = C^2H \cong \mathbb{Z}$, while $Z(G) \cong \mathbb{Z}^2$.

Nilpotent Groups: properties

Lemma

1. *Every subgroup of a nilpotent group is nilpotent.*
2. *If G is nilpotent and $N \lhd G$ then G/N is nilpotent.*
3. *The direct product of a family of nilpotent groups is again nilpotent.*

NB (3) not true for semidirect products. Not even for $\mathbb{Z}^n \rtimes \mathbb{Z}$.

Nilpotent Groups: a key property

Theorem

Every subgroup H of a finitely generated nilpotent group G is finitely generated.

Proof by induction on the class of nilpotency k of G.

For $k = 1$, G is abelian finitely generated. Assume the assertion true for k.

Let G be nilpotent of class $k + 1$, let $H \leqslant G$.

By the inductive assumption, $H_1 = H \cap C^2G$ is finitely generated.

$H_2 = H/(H \cap C^2G)$ is finitely generated because subgroup of G/C^2G, abelian finitely generated.

Thus, H fits into the short exact sequence

$$1 \to H_1 \to H \xrightarrow{\pi} H_2 \to 1,$$

where H_1, H_2 are finitely generated.

Therefore H is also finitely generated. □

Nilpotent Groups

We generalize $\left[C^iG, G\right] = C^{i+1}G$ to: the lower central series is graded, that is

Proposition

For every $i, j \geqslant 1$
$$\left[C^iG, C^jG\right] \leqslant C^{i+j}G\,. \tag{1}$$

First, recall that $[a, b]^{-1} = [b, a]$, whence $[A, B] = [B, A]$.

Lemma

If A, B, C *normal subgroups in* G*, then* $[A, B, C] \lhd G$ *and it is generated by* $[a, b, c]$ *with* $a \in A, b \in B, c \in C$*.*

Generation of $[A, B, C]$

Proof. $[A, B, C] \lhd G$ follows from $[x, y]^g = [x^g, y^g]$.

$[A, B, C]$ generated by $[k, c]$, $c \in C$, k product of n commutators $[a, b]$ or inverses.

We prove, by induction on n, that $[k, c]$ is a product of finitely many $[a, b, c]$ and inverses.