Infinite Groups

Cornelia Druțu

University of Oxford

Part C course MT 2024, Oxford

Nilpotent Groups

We generalize $\left[C^i G, G\right] = C^{i+1} G$ to: the lower central series is graded, that is

Proposition

For every $i, j \geqslant 1$
$$\left[C^i G, C^j G\right] \leqslant C^{i+j} G\,. \quad (1)$$

First, recall that $[a, b]^{-1} = [b, a]$, whence $[A, B] = [B, A]$.

Lemma

If A, B, C *normal subgroups in* G*, then* $[A, B, C] \lhd G$ *and it is generated by* $[a, b, c]$ *with* $a \in A, b \in B, c \in C$.

Generation of $[A, B, C]$

Proof. $[A, B, C] \lhd G$ follows from $[x, y]^g = [x^g, y^g]$.

$[A, B, C]$ generated by $[k, c]$, $c \in C$, k product of n commutators $[a, b]$ or inverses.

We prove, by induction on n, that $[k, c]$ is a product of finitely many $[a, b, c]$ and inverses.

$n = 1$: consider the case $[t^{-1}, c]$, where $t = [a, b]$.

$[t^{-1}, c] = [c, t]^{t^{-1}} = [c^{t^{-1}}, t] = [c', t] = [t, c']^{-1} = [a, b, c']^{-1}$.

Generation of $[A, B, C]$ 2

Assume the statement is true for n, let $k = k_1 t$, where t is $[a, b]$ or $[a, b]^{-1} = [b, a]$, and k_1 product of n commutators.

$$[k_1 t, c] = [t, c]^{k_1}[k_1, c]\,.$$

Both $[t, c]^{k_1}$ and $[k_1, c]$ are products of commutators $[a, b, c]$ and inverses, by the induction assumption and the fact that A, B, C are normal subgps. $\square$

Exercise

Prove the same result for $[H_1, \ldots, H_n]$, where all H_i are normal subgroups of G.

Second Key Lemma

Lemma

Assume that A, B, C are normal subgroups in G. Then

$$[A, B, C] \leqslant [B, C, A][C, A, B]\,. \tag{2}$$

Proof Uses the previous Lemma and the Hall identity:

$$\left[x^{-1}, y, z\right]^{x} \left[z^{-1}, x, y\right]^{z} \left[y^{-1}, z, x\right]^{y} = 1\,. \tag{3}$$

The latter identity implies

$$[a, b, c]^{a^{-1}} \leqslant [B, C, A][C, A, B].$$

□

Proof of Proposition

We prove by induction on $i \geqslant 1$ that for every $j \geqslant 1$,

$$\left[C^i G, C^j G\right] \leqslant C^{i+j} G\,. \tag{4}$$

For $i = 1$: definition of $C^{j+1} G$.

Assume true for i and prove for $i + 1$.

Consider $j \geqslant 1$ arbitrary.

$$[C^{i+1} G, C^j G] = [C^i G, G, C^j G] \leqslant [G, C^j G, C^i G][C^j G, C^i G, G] \leqslant$$

$$[C^{j+1} G, C^i G][C^{j+i} G, G] = [C^i G, C^{j+1} G] C^{j+i+1} G \leqslant C^{j+i+1} G\,,$$

since $[C^i G, C^{j+1} G] \leqslant C^{j+i+1} G$ by the inductive assumption.

□

Back to a more global picture

We study the following classes of groups:

Nilpotent finitely generated $\subset$ Polycyclic $\subset$ Solvable finitely generated

Definition

Given a class $\mathcal{X}$ of groups, a group G is said to be poly-$\mathcal{X}$ if it admits a subnormal descending series:

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_k \triangleright G_{k+1} = \{1\},$$

such that each G_i/G_{i+1} belongs to the class $\mathcal{X}$.

Polycyclic if $\mathcal{X} =$ all cyclic groups.

Poly-C_∞ if $\mathcal{X} = \{\mathbb{Z}\}$.

Solvable if $\mathcal{X} =$ all abelian groups.

Differences and similarities

Nilpotent finitely generated groups= the only groups with polynomial growth.

Polycyclic (hence also nilpotent f.g.) groups = finitely presented, linear (therefore residually finite) while solvable groups are not necessarily finitely presented, linear or residually finite.

A different behaviour of the torsion:

$$\mathrm{Tor}\, G = \{g \in G \mid \exists n \geq 1 \text{ s.t. } g^n = 1\}.$$

When G nilpotent, $\mathrm{Tor}\, G$ is a characteristic subgroup of G.

When G nilpotent and moreover f.g., $\mathrm{Tor}\, G$ is a finite characteristic subgroup of G.

When G polycyclic, $\mathrm{Tor}\, G$ not necessarily a subgroup of G, nor a finite subset of G. Examples in Ex. Sheet 2.

Torsion for nilpotent groups

Theorem

When G is nilpotent (not necessarily finitely generated), $\mathrm{Tor}\, G$ is a characteristic subgroup.

Proof by induction on the nilpotency class.

A key result for this induction:

Lemma

Let G be nilpotent of class k. For every $x \in G$ the subgroup H generated by x and C^2G is a normal subgroup, nilpotent of class $\leqslant k-1$.

Proof that $H = \langle x, C^2G \rangle$ of class $\leqslant k-1$.

Since $C^2G \lhd G$,

$$H = \{x^m c \mid m \in \mathbb{Z},\, c \in C^2G\}.$$

H normal: $\forall g \in G$, and $h \in H$, $h = x^m c$, $ghg^{-1} = x^m[x^{-m}, g]gcg^{-1}$.
The last two factors are in $C^2G \Rightarrow$ the product is in H.
We prove $C^2H \leqslant C^3G$ (implying H is of class $\leqslant k-1$).
Let $h = x^m c_1$, $h' = x^n c_2$ with $c_i \in C^2G$.

$$[h, h'] = [h, x^n c_2] = [h, x^n]\,[x^n, [h, c_2]]\,[h, c_2].$$

The last term is in C^3G, hence the middle term is in C^4G.
The first term can be rewritten as

$$[h, x^n] = [x^m c_1, x^n] = [x^m, [c_1, x^n]][c_1, x^n].$$

The last term is in C^3G and the first in C^4G. $\square$

Torsion for nilpotent groups

Theorem

When G is nilpotent (not necessarily finitely generated), $\mathrm{Tor}G$ is a characteristic subgroup.

Proof by induction on the nilpotency class, using:

Lemma

Let G be nilpotent of class k. For every $x \in G$, the subgroup H generated by x and C^2G is a normal subgroup, nilpotent of class $\leqslant k-1$.

For $k = 1$, G is abelian, statement immediate.