C3.3 Diff. mflds – class 1 30 October 2024

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set 3

X top. sp.

~~Chart~~ **Def'n** A chart is (U, ϕ), $U \subseteq \mathbb{R}^n$ open, $\phi: U \to X$ continuous, s.t. $\phi(U) \subseteq X$ open & $\phi: U \to \phi(U)$ is a homeomorphism.

Def'n An atlas is a family of mutually compatible charts whose union is X $\left(X = \bigcup \phi_i(U_i)\right)$; compatible means $\psi^{-1} \circ \phi : \underset{\cap\ \mathbb{R}^n}{\phi^{-1}(\psi(U))} \to \underset{\cap\ \mathbb{R}^n}{\psi^{-1}(\phi(U))}$ is smooth.

Given an atlas, $\exists$ unique maximal atlas containing it (easy to see why!)

Def'n A C^∞ mfld is a Hausdorff second countable X with a maximal atlas $\mathcal{A}$.

Part B 4. $X = \mathbb{R}^2 \setminus \{0\} / (x, y) \sim (\lambda x, \lambda^{-1} y)$.

with the quotient topology from $\pi: \mathbb{R}^2 \setminus \{0\} \to X$.

Charts: (U, ϕ), (V, ψ) $U = V = \mathbb{R}$,

$\phi(u) = [u, 1]$, $\psi(v) = [1, v]$.

Need to show:

- (U,ϕ), (V,ψ) are charts w/ image $\phi(U) = X \setminus \{[1,0]\}$ and $\psi(V) = X \setminus \{[0,1]\}$.
- Show compatibility by computing transition fn.

$\Rightarrow$ $\{(U,\phi), (V,\psi)\}$ is an atlas.

$\phi : \mathbb{R} \to X$ is actually

$$\phi : \mathbb{R} \to \mathbb{R}^2 \xrightarrow{\pi} X$$

$$u \mapsto (u,1) \mapsto [u,1]$$

both maps are continuous.

Clearly all elements of X are in $\phi(\mathbb{R})$ but $[1,0]$.

The inverse to ϕ is $\chi([x,y]) = xy$;

$$\chi : X \setminus \{[1,0]\} \to \mathbb{R}$$

note that for any $Y \subset \mathbb{R}$, $\chi^{-1}(Y) = (\pi \circ \varphi^{-1})(Y)$

when $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is $\varphi(x,y) = xy$.

But for Y open, $\varphi^{-1}(Y)$ is open as φ is continuous

$\Rightarrow$ $\chi^{-1}(Y)$ open since π is the quotient map.

Now, is $\phi(U) = X \setminus \{[1,0]\}$ open?

~~it is~~ $\psi^{-1}(X \setminus \{[1,0]\}) = \mathbb{R}^2 \setminus \{\mathbb{R} \times \{0\}\}$

open ✓

Transition function $\psi^{-1} \circ \phi : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$

why this domain? $\phi(U) = X \setminus \{[1,0]\}$

$\psi(V) = X \setminus \{[0,1]\}$

$\phi^{-1}(\psi(V)) = \psi^{-1}(\phi(U)) = \mathbb{R} \setminus \{0\}$

now $(\psi^{-1} \circ \phi)(x) = \psi^{-1}([x,1]) = \psi^{-1}([1,x])$

$= x$.

(smooth) with smooth inverse).

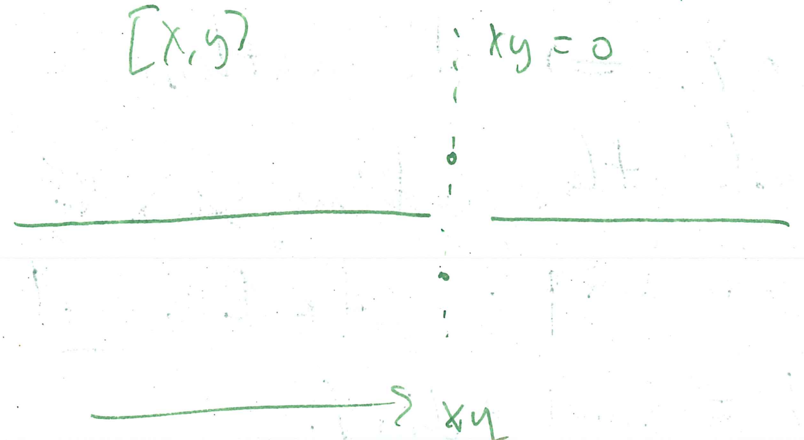
X is not Hausdorff! Consider $[0,0]$; $\phi^{-1}([0,1]) = 0$

any neighbourhood ~~is then~~ must then contain

some $[-\varepsilon, 1]$ and $[\varepsilon, 1]$; (charts are homeomorphisms)

similarly, any neigh. of $[1,0]$ must contain some

$[1,\delta]$; but $[1,\delta] = [\delta, 1]$..



line with two origins

5. X, Y mflds. Show that $X \times Y$ has a <u>unique</u> mfld structure s.t. if (U, ϕ), (V, ψ) are charts on X, Y, then $(U \times V, \phi \times \psi)$ is a chart on $X \times Y$. Then $\dim (X \times Y) = \dim X + \dim Y$.

To do this, we first build an atlas.

Note: if X, Y are Hausdorff & second countable, so is $X \times Y$.

Why is $(U \times V, \phi \times \psi)$ a chart?

$U \times V$ is open, $\phi(U) \times \psi(V)$ is open,
$= (\phi \times \psi)(U \times V)$

clearly this is a homeomorphism.

Mutual compatibility:

$$(\phi_1 \times \psi_1)^{-1} \circ (\phi_2 \times \psi_2) = (\phi_1^{-1} \circ \phi_2) \times (\psi_1^{-1} \circ \psi_2)$$

which is clearly smooth.

Now for each $(x, y) \in X \times Y$, if $x \in U$, $y \in V$, $(x, y) \in U \times V$

so for atlas $\mathcal{A}$ on X, $\mathcal{B}$ on Y,

$\mathcal{A} \times \mathcal{B}$ covers $X \times Y$. $\Rightarrow$ it's an atlas.

There is a unique maximal atlas containing $\mathcal{A} \times \mathcal{B}$, so the mfld structure is unique. $\mathcal{A} \times \mathcal{B}$ is <u>not</u> maximal!

dimension: $\dim U \times V = \dim U + \dim V$.

6. ~~(a)~~ X compact mfld, $\{U_i\}$ open cover,

(a) Show that $\forall x \in X$, $\exists\, i_x \in I$, $\xi_x : X \to \mathbb{R}$, $\xi_x \geq 0$, $\xi_x(x) > 0$, $\operatorname{supp} \xi_x \subseteq U_{i_x}$.

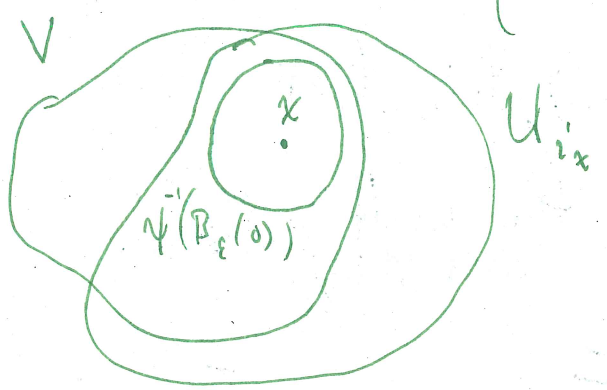
proof. First, let (V, ψ) be a chart in a neigh. V of x, so that $\psi^{-1}(x) = \vec{0} \in \mathbb{R}^n$. Let $\varepsilon > 0$ be s.t.

$$\overline{B_\varepsilon(0)} \subset \psi^{-1}(V \cap U_{i_x})$$

Let $\tilde{\xi}(\vec{y}) = \begin{cases} e^{-\frac{1}{\varepsilon^2 - |y|^2}} & \text{for } |y| \leq \varepsilon \\ 0 & \text{o.w.} \end{cases}$ Then $\tilde{\xi} : \mathbb{R}^n \to \mathbb{R}$ is clearly C^∞ away from $\partial \overline{B_\varepsilon(0)}$; in fact it is smooth even there as all partial derivatives vanish (can check).

Let $\xi_x : X \to \mathbb{R}$ be defined by

$$\xi_x(z) = \begin{cases} \tilde{\xi}(\psi^{-1}(z)) & \text{if } z \in V \\ 0 & \text{otherwise.} \end{cases}$$



Then clearly

$$\xi_x(x) = \tilde{\xi}(\psi^{-1}(x)) = \tilde{\xi}(0) = e^{-\frac{1}{\varepsilon^2}} > 0,$$

clearly $\xi_x \geq 0$, and $\operatorname{supp} \xi_x = \psi(\overline{B_\varepsilon(0)}) \subseteq U_{i_x}$ by assumption.

(b) Let $V_x = \{ g \in X \mid g_x(g) > 0 \}$. Show that $\exists\ S \subseteq X$ finite s.t. $X = \bigcup_{x \in S} V_x$.

proof. V_x is open as ~~it is the prei~~ $V_x = g_x^{-1}((0,\infty))$ and g_x is continuous. Clearly $x \in V_x$; so $X = \bigcup_{x \in X} V_x$; so by compactness the result follows.

(c) Show that $\sum_{x \in S} g_x : X \to \mathbb{R}$ is > 0.

For each $i \in I$, let $\eta_i = \left(\sum_{x \in S} g_x \right)^{-1} \sum_{x \in S \mid i_x = i} g_x$.

Show that this is a part. of unity.

proof. $\sum_{x \in S} g_x \geq 0$ is obvious; also for each $g \in X$, by (b), $g \in V_{x'}$ for some x', so by def'n $g_{x'} > 0$

so $\sum_{x \in S} g_x(g) \geq g_{x'}(g) > 0$.

Now it suffices to show (i), (ii), (iii), (iv).

(i): $\{ x \in S \mid i_x = i \} \subseteq S$ so the result (inequalities) follows.

(ii) $\operatorname{supp} \eta_i \subset \bigcup_{x \in S \mid i_x = i} \operatorname{supp} g_x \subset \bigcup_{i_x = i} \operatorname{supp} g_x \subset U_i$ by (a)

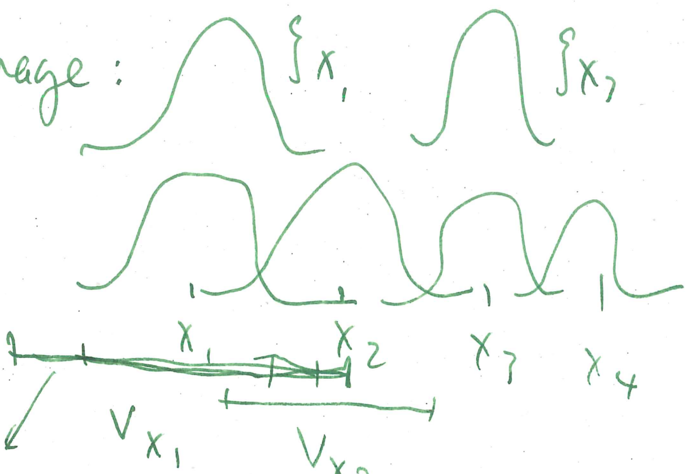
(iii) $\eta_i = 0$ unless $\{x \in S \mid i_x = i\} \neq \emptyset$.

But S is finite, and so is $\{i_x \mid x \in S\}$, so only finitely many $i \in I$ exist s.t. $\eta_i \neq 0$. Local finiteness follows.

(iv) $$\sum_{i \in I} \eta_i = \Big(\sum_{x \in S} \rho_x\Big)^{-1} \sum_{i \in I} \sum_{\{x \in S \mid i_x = i\}} \rho_x$$

$$= \Big(\sum_{x \in S} \rho_x\Big)^{-1} \sum_{x \in S} \rho_x = 1.$$

Image:



: $\sum_{x \in S} \rho_x$

$U_{i_{x_1}}$ (for instance).

There may be many $x_j \in X$ s.t. $\operatorname{supp} \rho_{x_j} \subset U_i$; we sum them together, as the open cover $\{U_i \mid i \in I\}$ meets given a point.