C3.3 Differentiable manifolds – Class 2 / Sheet 2 13 Nov Set 3

3 (a). Let $S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} |x_j|^2 = 1\}$

Explain why $T_{(x_0,\dots,x_n)} S^n \cong \{(y_0, \dots, y_n) \in \mathbb{R}^{n+1} / x_0 y_0 + \dots + x_n y_n = 0\}$

proof. We first recall our definitions

Def. Let X be a mfld. A tangent vector at $x \in X$ is a linear map $v : C^\infty(X) \to \mathbb{R}$ satisfying $v(ab) = a(x)v(b) + b(x)v(a)$ (Leibniz rule).

Set of tangent vectors at a point $x \in X$: $T_x X \cong \mathbb{R}^n$, $n = \dim X$.

Def. Given $f : X \to Y$ smooth, define the differential $T_x f : T_x X \to T_{f(x)} Y$ by $(T_x f)(v) : a \mapsto v(a \circ f)$ for $v \in T_x X$, $a \in C^\infty(Y)$.

Prop. if $g : Y \to Z$, $T_x(g \circ f) = T_{f(x)} g \circ T_x f : T_x X \to T_z Z$. $z = g(f(x))$

covariant functoriality.

Def. An immersion f satisfies $T_x f : T_x X \to T_{f(x)} Y$ injective $\forall x$

submersion surjective

embedding if it is an immersion & $f : X \to f(X)$ is a homeomorphism, $f(X) \subset Y$ the subspace topology.

Recall that $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an embedding, consequently an immersion. We have a map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $f(x) = f((x_0, \dots, x_n)) = x_0^2 + \dots + x_n^2 - 1$.

We would like to compute $T_x f : C^\infty(\mathbb{R}) \to \mathbb{R}$.

Let's do this in full generality. This is the last time you'll see explicit charts, I promise...

Let $f : X^n \to Y^m$ be smooth, $x \in X$, $\varphi : U \to X$ a chart in a neigh. of x and $\psi : V \to Y$ a chart in a neigh. of $f(x)$. Say $\varphi(\vec{u}) = x$ and $\psi(\vec{v}) = f(x)$. We know that $\{\partial_j^X, \ j = 1, \dots, n\}$ are a basis for $T_x X$, where $\partial_j^X : C^\infty(X) \to \mathbb{R}$ is given by $\partial_j^X(b) = \frac{\partial}{\partial x_j}(b \circ \varphi)\Big|_{\vec{u}}$

similarly, basis $\partial_j^Y(a) = \frac{\partial}{\partial y_j}(a \circ \psi)\Big|_{\vec{v}}$ for $T_{f(x)}Y$.

We also choose coordinates $y_1, \dots, y_m$ on V; now compute $T_x f$: by def'n $(T_x f)(v) : a \longmapsto v(a \circ f)$, where $a \in C^\infty(Y)$.

Take $v = \partial_j^X$; Then $[(T_x f)(\partial_j^X)](a)$

$$= \partial_j^X(a \circ f) = \frac{\partial}{\partial x_j}(a \circ f \circ \varphi)\Big|_{\vec{u}} = \frac{\partial}{\partial x_j}\Big((a \circ \psi) \circ (\psi^{-1} \circ f \circ \varphi)\Big)\Big|_{\vec{u}}$$

Now, write f in coordinates, ensure explicitly $\psi^{-1} \circ f \circ \varphi : \mathbb{R}^n \to \mathbb{R}^m$ as $(\psi^{-1} \circ f \circ \varphi)(x_1, \dots, x_n) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix}$.

We now use the multivariable chain rule $\mathbb{R}^n \xrightarrow{\psi^{-1}\circ f\circ\varphi} \mathbb{R}^m \xrightarrow{(a\circ\psi)} \mathbb{R}$ to get

$$= \sum_{i=1}^{m} \frac{\partial}{\partial x_j}(\psi^{-1}\circ f\circ\varphi)_i\Big|_{\vec u} \cdot \frac{\partial}{\partial y_i}(a\circ\psi)\Big|_{(\psi^{-1}\circ f\circ\varphi)(\vec u)} .$$

However, $\varphi(\vec u)=x$, $f(x)=\psi(\vec v)$, so $(\psi^{-1}\circ f\circ\varphi)(\vec u)=\vec v$ and this is consequently

$$= \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(\vec u)\, \frac{\partial}{\partial y_i}(a\circ\psi)\Big|_{\vec v} = \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(\vec u)\, \partial_i^{Y}\right) a$$

where $\{\partial_i^{Y} \mid i=1,\dots,n\}$ span $T_{f(x)}Y$ via $\partial_i^{Y} a = \frac{\partial}{\partial y_i}(a\circ\psi)\Big|_{\vec v}$.

Therefore the general formula is

$$T_x f\left(\sum_{j=1}^{n} c_j\, \partial_j^{X}\right) = \sum_{i,j=1}^{m,n} c_j \frac{\partial f_i}{\partial x_j}(\vec u)\, \partial_i^{Y}.$$

Coming back to our problem, taking coordinate r on $\mathbb{R}$, (chart is identity)

$$(T_x f)(y_0,\dots,y_n) = T_x f\left(\sum_{j=0}^{n} y_j\, \partial_j\right) = \sum_{j=0}^{n} y_j \frac{\partial f}{\partial x_j}\Big|_x \frac{d}{dr}$$

$$= \left(\sum_{j=0}^{n} 2x_j y_j\right)\frac{d}{dr} = \sum_{j=0}^{n} 2x_j y_j,$$ where we identify $T_x\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$, $T_0\mathbb{R} \cong \mathbb{R}$.

Since $\vec x \neq 0$ on S^n, $T_x f$ is surjective.

Now we make use of the functoriality of the pushforward.

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[rr, bend left, "0"] & \mathbb{R}^{n+1} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$ satisfies $f\circ i = 0$.

$$(x_0,\dots,x_n) \longmapsto (x_0,\dots,x_n) \longmapsto x_0^2+\dots+x_n^2-1$$

Consequently there are the induced pushforwards

$$\begin{tikzcd} 0 \arrow[r] & T_x S^n \arrow[r, hook, "T_x i"] \arrow[rr, bend right] & T_x \mathbb{R}^{n+1} \arrow[r, "T_x f"] & T_0 \mathbb{R} \arrow[r] & 0 \end{tikzcd}$$

$$T_x f \circ T_x i = T_x(f \circ i) = T_x 0 = 0$$

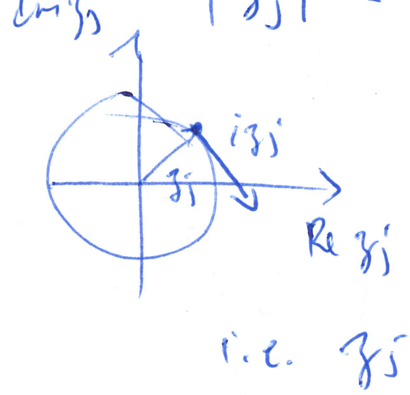
$T_x i$ is injective since i is an embedding; $T_x f$ is surjective by what we just showed. $\rightsquigarrow$ $\dim \ker(T_x f) = n$; since this is a chain complex with $\operatorname{im} T_x i \subseteq \ker T_x f$ and both are of dim n, we have that it is a short exact sequence of vector spaces & it is split, so that $\exists$ canonical isomorphism

$$T_x S^n \cong \ker(T_x f) = \{(y_0, \dots, y_n) \mid \sum_{j=0}^{n} x_j y_j = 0\}.$$

(b) Show that each odd-dimensional sphere has a nonvanishing vector field.

proof. The hint said $\mathbb{R}^{2k+2} \cong \mathbb{C}^{k+1}$; more specifically we can say $S^{2k+1} = \{(z_0, \dots, z_k) \in \mathbb{C}^{k+1} \mid |z_0|^2 + \dots + |z_k|^2 = 1\}$; projecting onto the j-th factor gives

$$|z_j|^2 = 1 - |z_0|^2 - \underset{\text{w/o } z_j \text{ term}}{\widehat{\cdots}} - |z_k|^2 ;$$ this is a circle.



You can pick a nonvanishing v.f. using the identification of (1) by $(-x_{2j+1}, x_{2j})$

i.e. $z_j = x_{2j} + i x_{2j+1} \longmapsto i z_j = -x_{2j+1} + i x_{2j}$.

One can now put all of these formed; each is nonvanishing where $z_j \neq 0$ and they are "orthogonal"; we get a vector field v given at $x = x_0, \dots, x_{2k+1}$ by

$$(-x_1, x_0, -x_3, x_2, \dots, -x_{2k+1}, x_{2k}).$$

<u>Claim</u>. There are no nonvanishing vector fields on S^{2n}, $n > 0$.

<u>proof</u>. The Poincaré–Hopf theorem says that for a v.f. v,

$$\sum_{x_i \text{ zeroes of } v} \operatorname{ind}_{x_i}(v) = \chi(M).$$

The Euler characteristic is defined as $\chi(M) = \sum \dim H_i(M, \mathbb{R}) (-1)^i$

We have $H_i(S^k) = \begin{cases} \mathbb{Z} & \text{if } i = 0, k \\ 0 & \text{otherwise} \end{cases}$; so

$$\chi(S^{2n}) = 2, \quad \chi(S^{2n+1}) = 0.$$

So, if v is nonvanishing everywhere, we get $0 = 2$, contradiction. $\square$

alternatively,

- rescale v to unit length
- flow it via the exponential map to length π
  - ↳ we get an antipodal map homotopic to id. but antipodal map has degree $(-1)^{n+1}$, since it is the composition of the $(n+1)$ plane reflections.

4. v, w vector fields on X^n.

$$[v,w] \equiv \sum_{i,j=1}^{n} \left(v_i \frac{\partial w_j}{\partial x_j} \frac{\partial}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j} \frac{\partial}{\partial x_i} \right)$$

Show that this is indep. of coordinates.

proof. First note that

$$v = \sum_j v_j \frac{\partial}{\partial x_j} = \sum_{a,j} v_j \frac{\partial y_a}{\partial x_j} \frac{\partial}{\partial y_a}$$

$$\equiv \sum_a v_a' \frac{\partial}{\partial y_a} \quad \text{w/}$$

$$w_b' = \sum_j w_j \frac{\partial y_b}{\partial x_j}$$

$$v_a' = \sum_i v_i \frac{\partial y_a}{\partial x_i}$$

Consequently

$$[v,w]' = \sum_{a,b} \left(v_a' \frac{\partial w_b'}{\partial y_a} \frac{\partial}{\partial y_b} - w_b' \frac{\partial v_a'}{\partial y_b} \frac{\partial}{\partial y_a} \right)$$

$$\cancel{= \sum_{a,b,i,j} \left(v_j \frac{\partial y_a}{\partial x_j} \right) \left[\frac{\partial w_j}{\partial y_a} \frac{\partial y_b}{\partial x_j} + w_j \frac{\partial^2 y_b}{\partial x_j \partial v} \right]}$$

$$= \sum_{a,b,i,j} v_i \frac{\partial y_a}{\partial x_i} \frac{\partial}{\partial y_a} \left(w_j \frac{\partial y_b}{\partial x_j} \right) \frac{\partial}{\partial y_b} - w_j \frac{\partial y_b}{\partial x_j} \frac{\partial}{\partial y_b} \left(v_i \frac{\partial y_a}{\partial x_i} \right) \frac{\partial}{\partial y_a}$$

$$= \sum_{b,i,j} v_i \frac{\partial}{\partial x_i} \left(w_j \frac{\partial y_b}{\partial x_j} \right) \frac{\partial}{\partial y_b} - \sum_{a,i,j} w_j \frac{\partial}{\partial x_j} \left(v_i \frac{\partial y_a}{\partial x_i} \right) \frac{\partial}{\partial y_a}$$

$$= \sum_{b,i,j} v_i \left(\frac{\partial w_j}{\partial x_i} \frac{\partial y_b}{\partial x_j} + w_j \frac{\partial^2 y_b}{\partial x_j \partial x_i} \right) \frac{\partial}{\partial y_b}$$

$$- \sum_{a,i,j} w_j \left(\frac{\partial v_i}{\partial x_j} \frac{\partial y_a}{\partial x_i} + v_i \frac{\partial^2 y_a}{\partial x_i \partial x_j} \right) \frac{\partial}{\partial y_a}$$

$$\overset{b \mapsto a}{=} \sum_{a,i,j} v_i \frac{\partial w_j}{\partial x_i} \frac{\partial y_a}{\partial x_j} \frac{\partial}{\partial y_a} - w_j \frac{\partial v_i}{\partial x_j} \frac{\partial y_a}{\partial x_i} \frac{\partial}{\partial y_a}$$

$$= \sum_{i,j} v_i \frac{\partial w_j}{\partial x_i} \frac{\partial}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j} \frac{\partial}{\partial x_i} = [v,w]$$ as required.

5. $\overset{u}{\varphi}_t(x_1,x_2) = (x_1+t, x_2)$, $\overset{v}{\varphi}_t(x_1,x_2) = (x_1, x_2+t)$

$$\overset{w}{\varphi}_t(x_1,x_2) = (\cos t\, x_1 + \sin t\, x_2, -\sin t\, x_1 + \cos t\, x_2)$$

Recall

$$v_x = \frac{\partial}{\partial t} \varphi_t(x) \Big|_{t=0} \quad \text{so}$$

$$\frac{d}{dt} \varphi^u_t \Big|_0 = (1,0) \; ; \quad u = \frac{\partial}{\partial x_1}$$

similarly $v = \frac{\partial}{\partial x_2}$, for w, it's $\begin{pmatrix} -\sin t\, x_1 + \cos t\, x_2 \\ -\cos t\, x_1 - \sin t\, x_2 \end{pmatrix}\Bigg|_{t=0}$

$$= \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, \text{ so } w = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}.$$

just compute the Lie brackets... intrinsic def'n

$$[X,Y] f = X(Y(f)) - Y(X(f)).$$

clearly $[u,v] = 0$

$$[u,w] = \frac{\partial}{\partial x_1}\left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}\right) - x_2 \frac{\partial^2}{\partial x_1^2} + x_1 \frac{\partial^2}{\partial x_2 \partial x_1}$$

$$= -x_1 \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial}{\partial x_2} + x_1 \frac{\partial^2}{\partial x_2 \partial x_1} = -\frac{\partial}{\partial x_2} = -v.$$

similarly for the other one. $[v,w]$.

6. A square matrix of size n, $v = \sum_{i,j} A_{ij} x_j \partial x_i$

integrate this into a 1-param. group of diffeomorphisms.

proof. Recall again $v_x = \frac{d}{dt} \varphi_t(x) \Big|_{t=0}$, write the diffeo. $\varphi_t(x)$ as $(x_1(t), \ldots, x_n(t)) = \varphi_t(x)$.

consequently, componentwise,

$$\sum_j A_{ij} x_j = \dot{x}_i(t);$$ this is well known to have general solution

$$x_i(t) = \sum_j \exp(At)_{ij} c_j;$$ this has

$$x_i(0) = id_{ij} c_j = c_i$$

$$\Rightarrow \quad \varphi_t(x) = \exp(At) x.$$