Infinite Groups

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Carl Friedrich Gauss

"I am coming more and more to the conviction that the necessity of our geometry cannot be demonstrated, at least neither by, nor for, the human intellect."

"We must admit with humility that, while number is purely a product of our minds, space has a reality outside our minds."

"Theory attracts practice as the magnet attracts iron."

Torsion for nilpotent groups

The torsion of a group G is:

$$\mathrm{Tor}\, G = \{g \in G \mid \exists n \geq 1 \text{ s.t. } g^n = 1\}.$$

Theorem

When G is nilpotent (not necessarily finitely generated), $\mathrm{Tor}\, G$ is a characteristic subgroup.

Proof by induction on the nilpotency class k, using:

Lemma

Let G be nilpotent of class k. For every $x \in G$, the subgroup H generated by x and $C^2 G$ is a normal subgroup, nilpotent of class $\leqslant k-1$.

For $k = 1$, G is abelian, statement immediate.

$\mathrm{Tor}\, G$ is a subgroup

Proof of Theorem continued.

Assume statement true for nilpotent groups of class $\leqslant k$, consider a $(k+1)$–step nilpotent group G.

For two elements a, b of finite order in G, we prove ab is of finite order.

$B = \langle b, C^2 G \rangle$ is nilpotent of class $\leqslant k$.

The inductive assumption $\Rightarrow \mathrm{Tor} B$ is a characteristic subgroup of B.

$B \lhd G \Rightarrow \mathrm{Tor} B \lhd G$.

Assume a is of order m.

Then

$$(ab)^m = aba^{-1}a^2ba^{-2}a^3b \cdots a^{-m+1}a^m ba^{-m}\,.$$

The right-hand side is a product of conjugates of $b \Rightarrow$ it is in $\mathrm{Tor} B$.

We conclude that $(ab)^m$ is of finite order. $\square$

Torsion of nilpotent f.g. groups

A torsion group G= all elements of finite order (i.e. $G = \mathrm{Tor}\, G$).

A torsion-free group G= a group with $\mathrm{Tor}\, G = \{1\}$.

Proposition

A finitely generated nilpotent torsion group is finite.

Proof by induction on the nilpotency class k.

For $k = 1$ it follows from the classification of f. g. abelian groups.

Assume true for nilpotent groups of class $\leqslant n$, consider G f. g. torsion group that is $(n+1)$–step nilpotent.

C^2G and G/C^2G are finite, by the inductive assumption, whence G finite. $\square$

Mal'cev's Theorem

Corollary

If G is nilpotent finitely generated then $\mathrm{Tor}\, G$ is a finite subgroup.

Proposition

If G is nilpotent then $G/\mathrm{Tor}\, G$ is torsion-free.

Theorem (Mal'cev)

Every finitely generated torsion-free nilpotent group G of class k embeds as a discrete subgroup in a simply-connected nilpotent Lie group L of class k, s.t. L/G compact.

Generalizes

- the case of finitely generated torsion-free abelian groups $\mathbb{Z}^n \leqslant \mathbb{R}^n$;
- the case of Heisenberg groups $H_{2n+1}(\mathbb{Z}) \leqslant H_{2n+1}(\mathbb{R})$.

Another theorem of Mal'cev

Question: Can the torsion/torsion-freeness be seen in the partial quotients of the upper/lower central series?

Theorem (Mal'cev)

Let G be a nilpotent group. The following are equivalent:

(a) $Z(G)$ is torsion-free;

(b) Each quotient $Z_{i+1}(G)/Z_i(G)$ is torsion-free;

(c) G is torsion-free.

Remark

The above characterization of "torsion-free" is not true if we replace the upper central series by the lower central series (Ex. Sheet 3).

Mal'cev's Theorem on torsion 2

(a)$\Rightarrow$ (b). By induction on the nilpotency class n of G. Clear for $n = 1$.
Assume true for nilpotent groups of class $< n$.
We first prove that the group $Z_2(G)/Z_1(G)$ is torsion-free.
We show that for each non-trivial $\bar{x} \in Z_2(G)/Z_1(G)$, there exists a homomorphism $\varphi\colon Z_2(G)/Z_1(G) \to Z_1(G)$ such that $\varphi(\bar{x}) \neq 1$.
Let $x \in Z_2(G)$ be an element which projects to $\bar{x} \in Z_2(G)/Z_1(G)$.Thus $x \notin Z_1(G)$, therefore there exists $g \in G$ such that $[g, x] \in Z_1(G) \setminus \{1\}$.
Define the map $\tilde{\varphi} : Z_2(G) \to Z_1(G)$ by:

$$\tilde{\varphi}(y) := [g, y].$$

Clearly, $\tilde{\varphi}(x) \neq 1$; $\tilde{\varphi}$ is a homomorphism (exercise).
Since $Z_1(G)$ is the center of G, $\tilde{\varphi}$ descends to a homomorphism $\varphi : Z_2(G)/Z_1(G) \to Z_1(G)$.
Since $Z_1(G)$ is torsion-free, $Z_2(G)/Z_1(G)$ is torsion-free.

Mal'cev's Theorem on torsion 3

We replace G by the group $\bar{G} = G/Z_1(G)$.
Since $Z_2(G)/Z_1(G)$ is torsion-free, the group $\bar{G}$ has torsion-free center.
By the inductive assumption, $Z_i(\bar{G})/Z_{i-1}(\bar{G})$ is torsion-free for every $i \geq 1$.

$$Z_i(\bar{G})/Z_{i-1}(\bar{G}) \cong Z_{i+1}(G)/Z_i(G),$$

for every $i \geqslant 1$.
Thus, every group $Z_{i+1}(G)/Z_i(G)$ is torsion-free, proving (b).
(b)$\Rightarrow$ (c). Let k be the nilpotency class, i.e. $G = Z_k(G)$.
$G = \bigsqcup_{i=1}^{k}[Z_i(G) \setminus Z_{i-1}(G)] \sqcup \{1\}$.
For each i, each $x \in Z_i(G) \setminus Z_{i-1}(G)$ and each $m \neq 0$ we have that $x^m \notin Z_{i-1}(G)$.
Thus $x^m \neq 1$.
Therefore, G is torsion-free. $\square$

Polycyclic groups

Definition

Let $\mathcal{X}$ be a class of groups.
G is poly-$\mathcal{X}$ if it admits a subnormal descending series:

$$G = N_0 \triangleright N_1 \triangleright \ldots \triangleright N_k \triangleright N_{k+1} = \{1\}, \tag{1}$$

such that each N_i/N_{i+1} belongs to $\mathcal{X}$, up to isomorphism.

Polycyclic if $\mathcal{X} =$ all cyclic groups.
Poly-C_∞ if $\mathcal{X} = \{\mathbb{Z}\}$.
Cyclic series of G= a series as in (1) with $\mathcal{X}$ set of cyclic groups. Its length is the number of non-trivial groups.
The length $\ell(G)$ of a polycyclic group is the least length of a cyclic series of G.
C_∞ series of G= a series as in (1) with $\mathcal{X} = \{\mathbb{Z}\}$.
By convention, $\{1\}$ is poly-C_∞.

Properties of polycyclic groups 1

Remark

1. *If G is poly-C_∞ then $N_i \simeq N_{i+1} \rtimes \mathbb{Z}$ for every $i \geqslant 0$; thus, the group G is obtained from $N_n \simeq \mathbb{Z}$ by successive semidirect products with $\mathbb{Z}$.*
2. *The above is no longer true for polycyclic groups (with $\mathbb{Z}$ replaced by "cyclic"). However: every polycyclic group contains a normal subgroup of finite index which is poly-C_∞ (proof to follow).*