Infinite Groups

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Proof by induction on the length of the series. □

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Proof by induction on the length of the series. $\square$

Corollary

A polycyclic torsion group is finite.

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Definition

A group is said to have property * virtually if some finite-index subgroup of it has the property *.

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2. *If $N \lhd G$, then G/N is polycyclic.*
3. *If $N \lhd G$ and both N and G/N are polycyclic then G is polycyclic.*
4. *Properties (1) and (3) hold with 'polycyclic' replaced by 'poly-C_∞', but not (2): $\mathbb{Z}_k$ is a quotient of $\mathbb{Z}$.*

Properties of polycyclic groups 2

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Proof. (1) and (2): See Section A in Ex. Sheet 3.
(3). Consider the cyclic series

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Given $\pi : G \to G/N$ and $H_i := \pi^{-1}(Q_i)$, a cyclic series for G is:

$$G \geqslant H_1 \geqslant \ldots \geqslant H_n = N = N_0 \geqslant N_1 \geqslant \ldots \geqslant N_k = \{1\}\,.$$

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- For every $k \geqslant 1$, $C^kG/C^{k+1}G$ is finitely generated abelian, hence there exists a finite subnormal descending series

$$C^kG = A_0 \geqslant A_1 \geqslant \cdots \geqslant A_n \geqslant A_{n+1} = C^{k+1}G$$

such that every quotient A_i/A_{i+1} is cyclic.

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Given any homomorphism $\varphi : \mathbb{Z}^n \to \mathrm{Aut}\,(\mathbb{Z}^m)$, the semidirect product $\mathbb{Z}^m \rtimes_\varphi \mathbb{Z}^n$ is poly-C_∞.

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Proof. Let $N = \langle X \mid r_1, \dots, r_k \rangle$, and $G/N = \langle \overline{Y} \mid \rho_1, \dots, \rho_m \rangle$ be finite presentations, where Y is a finite subset of G s. t. $\overline{Y} = \{yN \mid y \in Y\}$.

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G is generated by $S = X \cup Y$. S satisfies the following relations:

$$r_i(X) = 1\,,\ 1 \leqslant i \leqslant k\,,\ \rho_j(Y) = u_j(X)\,,\ 1 \leqslant j \leqslant m\,, \tag{1}$$

$$x^y = v_{xy}(X)\,,\ x^{y^{-1}} = w_{xy}(X)\,. \tag{2}$$

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We denote the above finite set of relations by T.

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Relations (2) imply that there exist a word $w_1(X)$ in X and a word $w_2(Y)$ in Y, such that $wK = w_1(X)w_2(Y)K$.

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Therefore $w_2(Y)$ is a product of finitely many conjugates of $\rho_i(Y)$, hence $w_2(Y)K$ is a product of finitely many conjugates of $u_j(X)K$, by (1).

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Then the image $\varphi(wK) = \varphi(v(X)K)$ is in N;

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$\varphi(v(X)K) = 1 \Leftrightarrow v(X)$ is a product of finitely many conjugates of $r_i(X)$.

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Then the image $\varphi(wK) = \varphi(v(X)K)$ is in N; therefore, $\varphi(v(X)K) = 1 \Leftrightarrow v(X)$ is a product of finitely many conjugates of $r_i(X)$.
This implies that $v(X)K = K$. $\square$

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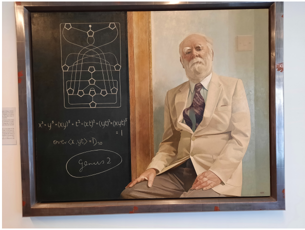

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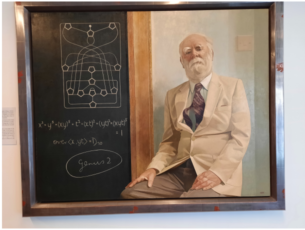

Theorem

Every finitely generated recursively presented group can be embedded as a subgroup of some finitely presented group.

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Proof. See Section A in Ex. Sheet 4.

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1. $$\bigcap_{i \in I} H_i = \{1\},$$

   where $\{H_i : i \in I\}$ is the set of all finite-index subgroups in G;

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Definition

A group satisfying the above is called residually finite.

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Assume $g \in \Gamma$ is a non-trivial element.

If g has a non-zero off-diagonal entry $g_{ij} \neq 0$, then $g_{ij} \neq 0 \mod p$, whenever $p > |g_{ij}|$.

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Indeed, we take subgroups $\Gamma(p) \leqslant \Gamma$, $\Gamma(p) = \ker(\varphi_p)$, where $\varphi_p : \Gamma \to GL(n, \mathbb{Z}_p)$ is the reduction modulo p.

Assume $g \in \Gamma$ is a non-trivial element.

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Theorem (Selberg's Lemma)

Let Γ be a finitely generated subgroup of $GL(n, F)$, where F is a field of characteristic zero. Then Γ contains a torsion-free subgroup of finite index.