Problem Sheet 2: Solutions

1. From the vertical component of the momentum equation (after applying the lubrication approximation) we have that
$$\frac{\partial p}{\partial z} = -\rho g$$
so that the pressure distribution within the drop is hydrostatic, i.e.
$$p = p_0 + \rho g\,[h(r,t) - z] \tag{1}$$
where we have neglected the pressure jump due to surface tension and p_0 is the constant atmospheric pressure.

   The horizontal component of the momentum equation is, in the lubrication approximation,
$$\mu \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial r} = \rho g \frac{\partial h}{\partial r}.$$
Integrating twice subject to $u(z=0) = 0$ (no-slip) and $u_z(z=h) = 0$ (no stress) we find that
$$u = \frac{\rho g}{2\mu} z(z - 2h) \frac{\partial h}{\partial r},$$
and
$$\bar{u} = -\frac{\rho g}{3\mu} h^2 \frac{\partial h}{\partial r}.$$
At this point is acceptable to quote the general result that
$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{q} = 0$$
where $\mathbf{q} = h\bar{u}\mathbf{e}_r$ is the fluid flux, though it would be reasonable to ask for this to be derived:
$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r \int_0^h u \,\mathrm{d}z\right) &= \int_0^h \frac{1}{r}\frac{\partial}{\partial r}(ru)\;\mathrm{d}z + h_r u(h) \\
&= -\int_0^h \frac{\partial w}{\partial z}\;\mathrm{d}z + u(h)h_r && [\text{Using } \nabla\cdot\mathbf{u} = 0] \\
&= -w(h) + w(0) + u(h)h_r \\
&= -\left[h_t + u(h)h_r\right] + u(h)h_r && \text{Using the k.b.c. and } w(0) = 0 \\
&= -h_t.
\end{aligned}$$

Thus we have

$$h_t = \frac{\rho g}{3\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right), \tag{2}$$

as desired.

We are told that the volume of the drop is a given constant V and so we must have

$$V = \int_0^{a(t)} 2\pi rh \; \mathrm{d}r.$$

In scaling terms we may write volume conservation as $V \sim R^2 H$ where R is a typical radial scale and H a typical vertical scale at time T. In scaling terms the governing pde (2) reads

$$\frac{H}{T} \sim \frac{\rho g}{\mu}\frac{H^4}{R^2}$$

from which we have

$$T \sim \frac{\mu}{\rho g} R^2 \left(H^{-3} \sim R^6/V^3\right).$$

Hence the typical radial scale R at time T must scale according to

$$R \sim \left(\frac{\rho g V^3}{\mu} T\right)^{1/8}$$

so that the radius $a(t)$ of the droplet must scale in the same way.

To progress further we non-dimensionalize lengths using $V^{1/3}$ and time using $\mu/\rho g V^{1/3}$ so that we wish to solve

$$h_t = \frac{1}{3r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right), \tag{3}$$

subject to the volume constraint

$$\int_0^{a(t)} rh \; \mathrm{d}r = 1/(2\pi). \tag{4}$$

Based on the scalings discussed above it is natural to seek a similarity solution of the form $h(r,t) = t^{-1/4}\Theta(\eta)$ where $\eta = rt^{-1/8}$ and we expect the drop to occupy the region $0 \le \eta \le \eta_*$ where $\eta_* = a(t)/t^{1/8}$ is the position of the edge of the drop; i.e. $h(a(t),t) = 0$.

[*Note that it is better to use the above similarity ansatz than the alternative t/r^8 since we take more spatial derivatives than time derivatives. However, this alternative approach does work if one is sufficiently careful.*]

Substituting this similarity form into (3) we have

$$-\frac{1}{8}(2\Theta + \eta\Theta') = \frac{1}{3\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\Theta^3\Theta'\right).$$

This may be integrated once to give

$$-\frac{3}{8}\eta^2\Theta = \eta\Theta^3\Theta'$$

where the constant of integration must vanish to ensure that the solution is well-behaved as $\eta \to 0$. A further integration gives

$$\Theta = \left(\frac{3}{4}\right)^{2/3}(\eta_*^2 - \eta^2)^{1/3}$$

where we have applied the boundary condition that $\Theta(\eta_*) = 0$.

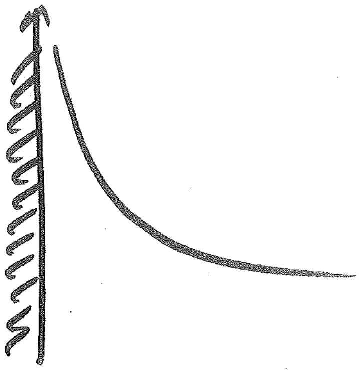
To determine the value of η_* we return to the similarity form of (4), which reads

$$\begin{aligned}\frac{1}{2\pi} &= \int_0^{\eta_*} \eta\Theta \ \mathrm{d}\eta = \left(\frac{3}{4}\right)^{2/3}\int_0^{\eta_*} \eta(\eta_*^2 - \eta^2)^{1/3} \ \mathrm{d}\eta \\ &= \left(\frac{3}{4}\right)^{2/3}\frac{3}{8}\eta_*^{8/3}\end{aligned}$$

from which we immediately have $\eta_* = (2^{10}/3^5\pi^3)^{1/8}$ and the result for $a(t)$ follows.

Q2

u(t)U



Lectures, Speed U Here Speed $Uu(t)$

$$\bar{x} = x - t \quad \longrightarrow \quad \bar{x} = x - \int_0^t u(t')\,dt'$$

$$\frac{\partial}{\partial \bar{t}} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \quad \longrightarrow \quad \frac{\partial}{\partial t} + u(t)\frac{\partial}{\partial x}$$

$$0 = h_t + h_z + \dots \quad \longrightarrow \quad 0 = h_t + u(t)h_z + \frac{\partial}{\partial z}\left[\frac{h^3}{3Ca}\left(h_{zzz} - Bo\right)\right]$$

Has scaled $\dfrac{\partial h^{\text{physical}}}{\partial t^{\text{physical}}}$ with $W = \delta U$
$= \dfrac{HU}{L}$.

But will scale with $\dfrac{H}{T_{\text{boundary}}}$, T_{boundary} = timescale boundary changes

$$= \frac{\cancel{U}|u(t)|}{\cancel{U}|\dot{u}(t)|}$$

$$\therefore \left(\frac{H}{T_{\text{boundary}}}\frac{1}{W}\right)\frac{\partial h}{\partial t} + \dots$$

$T \to \infty$ as $\dot{u} \to 0$

$$\frac{H}{T_b}\frac{L}{HU} = \underline{\underline{\left(\frac{L}{T_b U}\right)}}$$

Rescalings, as in notes

$$\lambda \frac{\partial \bar{h}}{\partial \bar{t}} + u(t)\frac{\partial \bar{h}}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}}\left[\bar{h}^3/3\left(\bar{h}_{\bar{z}\bar{z}\bar{z}} - \overbrace{\text{Bo}}^{\delta^2 \ll 1}\right)\right] = 0$$

λ: $\frac{L}{T_b U} \ll 1$ for $\dot{u}$ sufficiently small.

$$\therefore\ u(t)\bar{h}_{\bar{z}} + \partial/\partial\bar{z}\left(\bar{h}^3/3\ \bar{h}_{\bar{z}\bar{z}\bar{z}}\right) = 0.$$

Proceeding...

$$u(t)\frac{\partial \tilde{h}}{\partial \tilde{z}} + \frac{\partial}{\partial \tilde{z}}\left(\frac{\tilde{h}^3}{3}\tilde{h}_{\tilde{z}\tilde{z}\tilde{z}}\right) = 0,$$

which integrates to give

$$u(t)\tilde{h} + \frac{\tilde{h}^3}{3}\tilde{h}_{\tilde{z}\tilde{z}\tilde{z}} = f(t).$$

As $\tilde{z} \to \infty$, we expect that $\tilde{h} \to \tilde{h}_0$, a constant and so

$$u(t)\tilde{h} + \frac{\tilde{h}^3}{3}\tilde{h}_{\tilde{z}\tilde{z}\tilde{z}} = u(t)\tilde{h}_0 \tag{6}$$

subject to $\tilde{h} \sim \tilde{z}^2/\sqrt{2}$ as $\tilde{z} \to -\infty$.

Following the lecture notes, we let $\tilde{h} = \tilde{h}_0 g(\zeta)$, $\tilde{z} = z_*\zeta$ so that (6) becomes

$$g + \frac{\tilde{h}_0^3}{u(t)z_*^3}\tfrac{1}{3}g^3 g_{\zeta\zeta\zeta} = 1.$$

We choose $z_* = \tilde{h}_0/u(t)^{1/3}$ so that we obtain the Landau–Levich equation

$$g + \tfrac{1}{3}g^3 g_{\zeta\zeta\zeta} = 1$$

with boundary conditions $g \to 1$ as $\zeta \to \infty$ and

$$g \sim \frac{\tilde{h}_0}{u(t)^{2/3}}\frac{\zeta^2}{\sqrt{2}}$$

as $\zeta \to -\infty$.

From the notes, we recall that the numerical solution of the Landau–Levich equation has $g \sim 0.67\zeta^2$ as $\zeta \to -\infty$ and so we have

$$\frac{\tilde{h}_0}{\sqrt{2}u(t)^{2/3}} = 0.67$$

and so $\tilde{h}_0 \approx 0.95u(t)^{2/3}$, as required.

3 We are given that

$$h_t + \left[\frac{h^3}{3\mathrm{Ca}}\left(h_{xxx} - \mathrm{Bo}\ h_x\right)\right]_x = 0.$$

Letting $h(x,t) = h_0 + \delta h_1(x,t)$ with h_0 constant we find that

$$0 = \delta\frac{\partial h_1}{\partial t} + \left[\frac{h_0^3 + 3\delta h_0^2 h_1}{3\mathrm{Ca}}\left(\delta h_{1,xxx} - \delta\ \mathrm{Bo}\ h_{1,x}\right)\right]_x + O(\delta^3).$$

Examining the $O(\delta)$ terms we immediately see that

$$0 = \frac{\partial h_1}{\partial t} + \frac{h_0^3}{3\mathrm{Ca}}\left(h_{1,xxxx} - \mathrm{Bo}\ h_{1,xx}\right),$$

as required.

Letting $h_1 = \Re\left[e^{\sigma t + ikx}\right]$ we have

$$\sigma = -\frac{h_0^3}{3\mathrm{Ca}}\left[(ik)^4 - \mathrm{Bo}(ik)^2\right] = -\frac{h_0^3}{3\mathrm{Ca}}\left(k^4 + \mathrm{Bo}\ k^2\right),$$

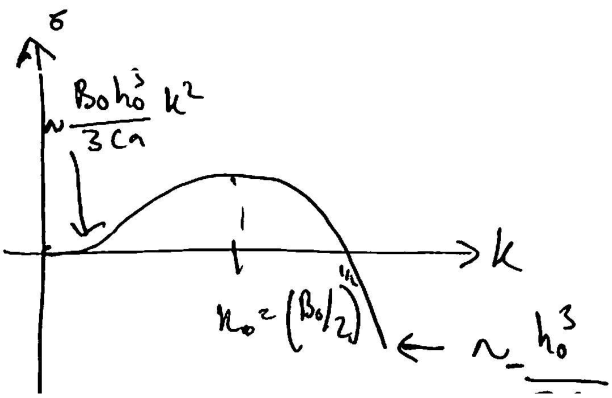
as required. Clearly $\sigma < 0$ for all $k \neq 0$; disturbances therefore decays with time and the film is stable to perturbations of any wavelength.

For $k = 0$, $\sigma = 0$ and so a uniform perturbation is neutrally stable — it neither grows nor decays.

When the film is beneath (rather than above) the plate, gravity acts in the opposite direction, Bo $\to -$Bo and so we have

$$\sigma = -\frac{h_0^3}{3\mathrm{Ca}}\left(k^4 - \mathrm{Bo}\ k^2\right),$$

A sketch is shown below:



We see that the surface tension term $(-k^4)$ is stabilising since it makes σ more negative, while gravity $(+\mathrm{Bo}\ k^2)$ is destabilising since it acts to increase σ.

The situation is unstable whenever $\sigma > 0$, i.e. for

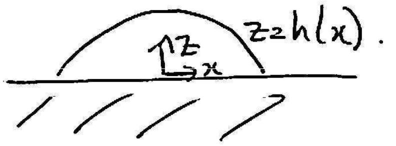
$$0 < k < \mathrm{Bo}^{1/2}.$$

The maximally unstable mode is that for which σ is maximised. We have

$$\sigma'(k) = -\frac{h_0^3}{3\mathrm{Ca}}(4k^3 - 2\mathrm{Bo}\ k)$$

and so the maximally unstable wavelength is $\lambda = 2\pi/k_*$ where $k_* = (\mathrm{Bo}/2)^{1/2}$.

Q4



The Kinematic boundary condition is:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = w - E \qquad (1)$$

In deriving lubrication flows for droplets used length R timescale R/U and velocity scale U.

Using these scales here, we find:

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{h}}{\partial \tilde{x}} = \tilde{w} - E/U$$

But U is thus far arbitrary, so we choose $U = E$

$$\Rightarrow \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = w - 1 \quad \text{on } z=h(x).$$

(This is consistent with scaling used for lubrication theory but have $Ca = \frac{\mu E}{\gamma}$ specified).

In the fluid we have the dimensionless lubrication equations:

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}$$

$$0 = \frac{\partial p}{\partial z} \Rightarrow p = f(x)$$

$$= -\frac{1}{Ca}\frac{\partial^2 h}{\partial x^2} \quad \text{on } z=h(x)$$

$$\Rightarrow p = -\frac{1}{Ca}\frac{\partial^2 h}{\partial x^2} \quad \text{everywhere.}$$

$$\therefore u = \frac{1}{2}\frac{\partial p}{\partial x} z(z-2h) \qquad \left[\because \begin{array}{l} u=0 \text{ on } z=0, \\ u_z=0 \text{ on } z=h \end{array}\right]$$

$$\Rightarrow h\bar{u} = \frac{1}{2}\frac{\partial p}{\partial x}\left[\frac{h^3}{3} - h^3\right] = -\frac{h^3}{3}\frac{\partial p}{\partial x} = \frac{h^3}{3Ca}\frac{\partial^3 h}{\partial x^3}.$$

Now:

$$\frac{\partial}{\partial x}\int_0^h u\,dz = h_x u(h) + \int_0^h \left(\frac{\partial u}{\partial x} = -\frac{\partial w}{\partial z}\right) dz$$

$$= h_x u(h) - \left[w(h) = 1 + \frac{\partial h}{\partial t} + h_x u(h)\right]$$

$$= -1 - \frac{\partial h}{\partial t}.$$

$$\therefore \quad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u} h\right) = -1$$

i.e. $$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{h^3}{3Ca}\frac{\partial^3 h}{\partial x^3}\right) = -1. \qquad (*).$$

$Ca \ll 1 \Rightarrow$ @ leading order:

$$\frac{\partial}{\partial x}\left(\frac{h_0^3}{3}\frac{\partial^3 h_0}{\partial x^3}\right) = 0 \Rightarrow \frac{\partial^3 h_0}{\partial x^3} = 0$$

$$\left[h_0^3 h_{0,xxx} = \text{cst} = 0 \quad \text{as } h_0 = 0 \text{ at } x = \pm s\right]$$

$$\Rightarrow h_0 = k(t)\left(s^2 - x^2\right)$$

To ensure $A(t) = \int_{-s}^{s} h_0\,dx = k(t) \cdot \frac{4s^3}{3}$

$$\Rightarrow k(t) = \frac{3}{4s^3} A(t).$$

Integrating the film eqn $(*)$ we have:

$$\frac{d}{dt}\int_{-s}^{s} h\,dx = -\int_{-s}^{s} 1\,dx = -2s$$

So, at leading order:

$$-2s = \frac{d}{dt}\int_{-s}^{s} h_0\,dx = \frac{d}{dt}A \Rightarrow \dot{A} = -2s, \text{ as desired.}$$

Let $h = h_0 + Ca\, h_1 + Ca^2 h_2 + \cdots$

and then calculate:

$$Ca\,\bar{u} = \frac{h^2}{3} h_{3x} = \frac{\cancel{h_0^2 h_{0,3x}}^{\,0}}{3} + Ca\left(\frac{h_0^2 h_{1,3x}}{3} + \frac{2h_0 h_1 h_{0,3x}}{3}\right) + \cdots$$

$$= \cancel{\bar{u}_0}^{\,0} + Ca\,\bar{u}_1 + \cdots$$

Sub this into evolution equation (*):

$$-1 = \frac{\partial h_0}{\partial t} + \frac{Ca}{Ca}\frac{\partial}{\partial x}\left(h_0 \bar{u}_1 + \underbrace{h_1 \bar{u}_0}_{0}\right) \quad 0 \text{ since } \bar{u}_0 = 0$$

$$\Rightarrow \quad h_0 \bar{u}_1 = -\int_0^x 1 + \frac{\partial h_0}{\partial t}\, dx' \qquad \left[\because \bar{u}_1(x=0) = 0 \text{ by symmetry}\right].$$

$$= -x - \int_0^x -\frac{3}{2s^2}\left(s^2 - x'^2\right) dx'$$

$$= -x + \frac{3}{2s^2}\left(s^2 x - \frac{x^3}{3}\right)$$

$$= \frac{x}{2s^2}\left(s^2 - x^2\right).$$

$$\Rightarrow \bar{u}_1 = \frac{\frac{x}{2s^2}\left(s^2 - x^2\right)}{\frac{3A(t)}{4s^3}\left(s^2 - x^2\right)}$$

$$\bar{u} = \bar{u}_1 + Ca\,\bar{u}_2 + \cdots$$

$$= \frac{2s}{3}\frac{x}{A(t)} + O(Ca) \qquad \text{as desired.}$$

If gravity is present then we have:

$$\frac{1}{3}h^2\left(h_{xxx} - B_0 h_x\right) = \bar{u}_0 + Ca\,\bar{u}_1 + \cdots$$

The analysis above will be unchanged provided that

$$h_{xxx} - B_0 h_x \simeq h_{xxx}$$

Now if we choose $L = S$ so that gradients of h_0 etc are all $O(1)$ then we require $B_0 \ll 1$

and hence: $\frac{\rho g}{\gamma} S^2 \ll 1$,

5. We have the usual lubrication equation for the horizontal velocity, u, i.e.

$$\mu \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}, \tag{7}$$

while at leading order the vertical component gives

$$\frac{\partial p}{\partial z} = -\rho g.$$

Integrating the latter and taking the pressure in the atmosphere to be 0, we have

$$p = \rho g \left[h(x,t) - z \right] - \gamma \frac{\partial^2 h}{\partial x^2}. \tag{8}$$

Integrating (7) we have that

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} z^2 + Az + B$$

where A and B are constants to be determined from the boundary conditions roblem:

- The condition of zero shear stress at the free surface gives

$$\left. \frac{\partial u}{\partial z} \right|_{z=h} = 0 \quad \implies A = -\frac{1}{\mu} \frac{\partial p}{\partial x} h.$$

- The slip condition $u = \lambda u_z$ at $z = 0$ gives

$$B = \lambda A = -\frac{\lambda}{\mu} \frac{\partial p}{\partial x} h$$

and hence

$$u = -\frac{1}{\mu} \frac{\partial p}{\partial x} \left[\tfrac{1}{2} z^2 - zh - \lambda h \right].$$

Now

$$\begin{aligned} \bar{u} &= \frac{1}{h} \int_0^h u \, \mathrm{d}z = \frac{1}{\mu h} \frac{\partial p}{\partial x} \left[\tfrac{1}{6} h^3 - \tfrac{1}{2} h^3 - \lambda h^2 \right]. \\ &= -\frac{1}{\mu} \frac{\partial p}{\partial x} \left[\tfrac{1}{3} h^2 + \lambda h \right]. \end{aligned}$$

From (8) we have that

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial h}{\partial x} - \gamma \frac{\partial^3 h}{\partial x^3}$$

and so

$$\bar{u} = \frac{\gamma}{\mu} \left(\frac{\partial^3 h}{\partial x^3} - \ell_c^{-2} \frac{\partial h}{\partial x} \right) \left(\tfrac{1}{3} h^2 + \lambda h \right),$$

where $\ell_c^2 = \gamma/\rho g$, as usual.

The associated thin film equation is eqn (2.27) from the notes and so we have

$$0 = \frac{\partial h}{\partial t} + \frac{\gamma}{\mu}\frac{\partial}{\partial x}\left[\left(\frac{\partial^3 h}{\partial x^3} - \ell_c^{-2}\frac{\partial h}{\partial x}\right)\left(\tfrac{1}{3}h^3 + \lambda h^2\right)\right]$$

[*It is worth emphasizing again that the general statement of conservation of mass (having used the kinematic boundary condition) is of the form*

$$\frac{\partial h}{\partial t} + \nabla\cdot\mathbf{q} = 0,$$

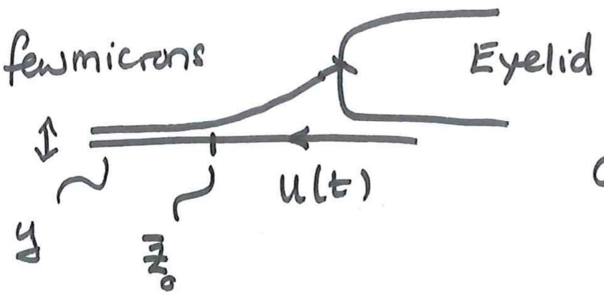
since this comes up frequently in such problems.]

Q6

From Q2

$$\eta\frac{\partial \bar{h}}{\partial t} + u(t)\frac{\partial \bar{h}}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}}\left[\frac{\bar{h}^3}{3}\left(\bar{h}_{\bar{z}\bar{z}\bar{z}} - \delta^2\right)\right] = 0$$

Dominant balance for thinnest possible film, A below as small as possible while there is still a balance



Let $y = A(\bar{z} - \bar{z}_0)$, $A \ll 1$

$$0 = \eta\frac{\partial \bar{h}}{\partial t} + uA\frac{\partial \bar{h}}{\partial y} + \frac{\partial}{\partial y}\left[\frac{\bar{h}^3}{3}\left(A^4\bar{h}_{yyy} - \delta^2 A\right)\right]$$

$A = \eta$, dominant balance $\quad 0 = \dfrac{\partial \bar{h}}{\partial t} + u(t)\dfrac{\partial \bar{h}}{\partial y}$

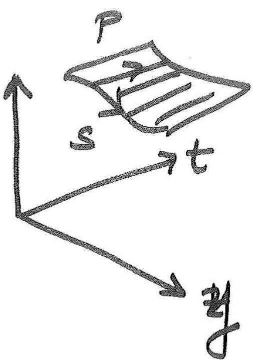
As $y \to 0$, $\quad \bar{h} \sim 0.95(u(t))^{2/3}$ $\therefore$ Take $\bar{h}(y=0,t) = 0.95(u(t))^{2/3}$.

$\therefore$ Noting $u(t) = e^{-t}$

$$\boxed{\begin{gathered} e^{t}\frac{\partial \bar{h}}{\partial t} + \frac{\partial \bar{h}}{\partial y} = 0 \\ h(0,t) = 0.95\, e^{-2t/3} \end{gathered}}$$

Cauchy data, on $p=0$.

$t = s, \quad y = 0, \quad h(s) = 0.95 e^{-2s/3}$



Characteristics

$$\frac{dt}{dp} = e^{t} \qquad \frac{dy}{dp} = 1 \qquad \frac{dh}{dp} = 0$$

$t(p=0) = s$	$y(p=0) = 0$	$h(p=0) = 0.95e^{-2s/3}$
$\therefore\ p = \int e^{-t}dt$	$y = p$	$h = 0.95e^{-2s/3}$
$p = -e^{-t} + e^{-s}$		

$$\therefore\ e^{s} = \frac{1}{p+e^{-t}} \qquad \therefore\ e^{-2s/3} = (p+e^{-t})^{2/3}$$

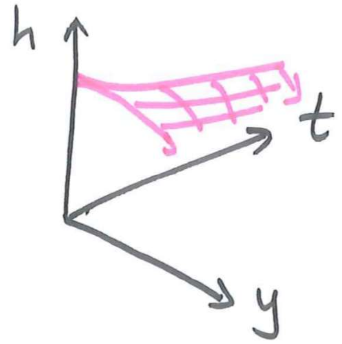
$$= (y+e^{-t})^{2/3}$$

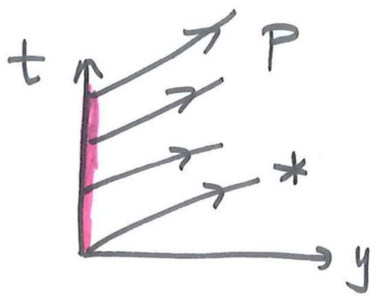
$y \to z$ to match question

$$\therefore \quad h = \left(z + e^{-t}\right)^{2/3} \cdot (0.95) \to 0.95\, z^{2/3}$$

as $t \to \infty$

Constraint on z





Limiting characteristic ($*$) corresponds to s=0 above

$$\frac{dy}{dt} = e^{-t}, \quad y = A - e^{-t} \quad \text{with } y = 0 \text{ when } t = 0$$

$$y = 1 - e^{-t}$$

$\therefore$ Range of $y \equiv z$ is $[0, 1)$.