Infinite Groups

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Life versus science

Albert Einstein: Politics is for the present, but an equation is something for eternity.

Mahatma Gandhi: Live as if you were to die tomorrow. Learn as if you were to live forever.

Residual finiteness

The idea: approximate an infinite group by its finite quotients.
So one needs to have enough finite quotients.

Proposition

Let G be a group. The following are equivalent:

1. $$\bigcap_{i \in I} H_i = \{1\},$$
   where $\{H_i : i \in I\}$ is the set of all finite-index subgroups in G;
2. *for every $g \in G \setminus \{1\}$, there exists a finite group Φ and a homomorphism $\varphi : G \to \Phi$, such that $\varphi(g) \neq 1$.*

Definition

A group satisfying the above is called residually finite.

Properties of RF

Proposition

1. *G, H residually finite (RF) $\Rightarrow G \times H$ RF;*
2. *G RF and $H \leqslant G \Rightarrow H$ RF;*
3. *$H \leqslant G$ of finite index and H RF $\Rightarrow G$ RF;*
4. *H finitely generated RF and Q RF $\Rightarrow H \rtimes Q$ RF.*

Remark

There exist short exact sequences

$$\{1\} \longrightarrow \mathbb{Z}_2 \stackrel{i}{\longrightarrow} G \stackrel{\pi}{\longrightarrow} Q \longrightarrow \{1\},$$

with Q finitely generated RF and G not RF (J. Millson 1979).

Corollary

The free group F_2 of rank 2 is residually finite. Every free group of (at most) countable rank is residually finite.

Remark

This in particular shows that G RF does not imply G/N RF, for $N \lhd G$.

Remark

Given a short exact sequence

$$\{1\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\pi} F(X) \longrightarrow \{1\},$$

with H finitely generated RF and X finite or countable, G is residually finite.

Back to polycyclic groups

Proposition

Polycyclic groups are finitely presented and residually finite.

Proof by induction on the length $\ell(G)$.

For $\ell(G) = 1$, G is cyclic.

Assume that the statement is true for polycyclic groups of length n, let G be polycyclic with $\ell(G) = n + 1$.

Let N_1 be the first (sub)normal subgroup in a cyclic series of minimal length $n + 1$.

Then N_1 is polycyclic of length n, hence finitely presented (respectively residually finite) by the induction hypothesis.

Induction proving polycyclic groups are FP and RF

We have the short exact sequence

$$\{1\} \longrightarrow N_1 \xrightarrow{i} G \xrightarrow{\pi} C \longrightarrow \{1\},$$

where C is cyclic.

This implies G finitely presented.

When C finite, N has finite index, hence G RF.

When $C = \mathbb{Z}$, $G = N_1 \rtimes \mathbb{Z}$, hence RF.

□

Normal poly-C_∞ subgroup

Proposition

A polycyclic group contains a normal subgroup of finite index which is poly-C_∞.

Proof By induction on the length $\ell(G) = n$.
For $n = 1$ the group G is cyclic and the statement true.
Assume the assertion is true for n and consider a polycyclic group G with a cyclic series of length $n + 1$.
The induction hypothesis implies that N_1 (the first group in the series) contains a normal subgroup S of finite index which is poly-C_∞.
Proposition 2.8, (2), in Revision Notes implies that S contains S_1 characteristic subgroup of N_1 of finite index.
Since $N_1 \lhd G$, S_1 is normal in G.
$S_1 \leqslant S \Rightarrow S_1$ is poly-C_∞.
If G/N_1 is finite then S_1 has finite index in G.

Normal poly-C_∞ subgroup 2

Assume G/N_1 is infinite cyclic.
Then the group $K = G/S_1$ contains the finite normal subgroup $F = N_1/S_1$ such that K/F is isomorphic to $\mathbb{Z}$.
In other words, we have the short exact sequence

$$\{1\} \longrightarrow F \xrightarrow{\varphi} K \xrightarrow{\psi} \mathbb{Z} \longrightarrow \{1\}\,.$$

Then K is a semidirect product of F and an infinite cyclic subgroup $\langle x \rangle$.
The conjugation by x defines an automorphism of F and since $\mathrm{Aut}(F)$ is finite, there exists r such that the conjugation by x^r is the identity on F.
We conclude that $\langle x^r \rangle$ is a finite index normal subgroup of K.
We have that $\langle x^r \rangle = G_1/S_1$, where G_1 is a finite index normal subgroup in G, and G_1 is poly-C_∞ since S_1 is poly-C_∞. □

Polycyclic torsion-free

Proposition

A polycyclic group contains a normal subgroup of finite index which is poly-C_∞.

Corollary

(a) *A poly-C_∞ group is torsion-free.*

(b) *A polycyclic group is virtually torsion-free.*

Proof. (a) Induction on the cyclic length. $n = 1 \Rightarrow G$ infinite cyclic.
Assume true for groups of cyclic length $\leq n$, let G with $\ell(G) = n + 1$ and N_1 first subgroup in a cyclic series of G.
Let g be an element of finite order in G.
Its image in $G/N_1 \simeq \mathbb{Z}$ is the identity, hence $g \in N_1$.
The induction assumption implies $g = 1$.
(b) follows from (a) and the Proposition. □

The Hirsch length

Proposition

The number of infinite quotients in a cyclic series of a polycyclic group G is the same for all series.
This number is called the Hirsch length of G, denoted by $h(G)$.

Proof uses the Jordan-Hölder Theorem:
Any two finite subnormal series in a group have equivalent refinements.
A series is a refinement of another series if the subgroups of the latter all occur in the former.
Two finite series are equivalent if they have the same sequence of quotients N_i/N_{i+1}, up to permutation.
To prove the proposition it then suffices to show the following

Lemma

A refinement of a cyclic series is also cyclic. Moreover, the number of quotients isomorphic to $\mathbb{Z}$ is the same for both series.

Proof of the lemma

Proof. Consider a cyclic series

$$H_0 = G \geqslant H_1 \geqslant \ldots \geqslant H_n = \{1\}\,.$$

A refinement of this series is composed of a concatenation of sub-series

$$H_i = R_k \geqslant R_{k+1} \geqslant \ldots \geqslant R_{k+m} = H_{i+1}\,.$$

H_i/H_{i+1} cyclic $\Rightarrow$ H_i/R_{j+1} cyclic (quotient) $\Rightarrow$ R_j/R_{j+1} cyclic (subgroup).
H_i/H_{i+1} finite $\Rightarrow$ all R_j/R_{j+1} are finite.
Assume $H_i/H_{i+1} \simeq \mathbb{Z}$.
By induction on $m \geqslant 1$: exactly one quotient $R_j/R_{j+1} \simeq \mathbb{Z}$.
For $m = 1$, clear. Assume true for m, consider the case of $m + 1$.
If H_i/R_{k+m} is finite then all R_j/R_{j+1} with $j \leqslant k + m - 1$ are finite.
R_{k+m}/R_{k+m+1} cannot be finite, otherwise H_i/H_{i+1} finite.
Therefore $R_{k+m}/R_{k+m+1} \simeq \mathbb{Z}$.

Proof of the lemma, continued

Assume $H_i/R_{k+m} \simeq \mathbb{Z}$.
Inductive assumption $\Rightarrow$ exactly one $R_j/R_{j+1} \simeq \mathbb{Z}, j \leqslant k+m-1$.
R_{k+m}/R_{k+m+1} is a subgroup of $H_i/R_{k+m+1} \simeq \mathbb{Z}$ such that the quotient by this subgroup is also isomorphic to $\mathbb{Z}$.
This can only happen when R_{k+m}/R_{k+m+1} is trivial. $\square$

Definition

Let G be a finitely generated nilpotent group of class k. Let m_i denote the free rank of the abelian group $C^iG/C^{i+1}G$. The Hirsch number of G is $h(G) = \sum_{i=1}^{k} m_i$.

Proposition

For each finitely generated nilpotent group the Hirsch number equals the Hirsch length.

Proof is Exercise 2, Ex. Sheet 3.