Infinite Groups

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Solvable groups

A first definition: poly-abelian is solvable.
We now provide a second definition.
$G' = [G, G]$ the derived subgroup of G.
The iterated commutator subgroups $G^{(k)}$ are defined inductively by:

$$G^{(0)} = G, G^{(1)} = G', \dots, G^{(k+1)} = \left(G^{(k)}\right)', \dots$$

All subgroups $G^{(k)}$ are characteristic in G.
The derived series of the group G is

$$G \trianglerighteq G' \trianglerighteq \dots \trianglerighteq G^{(k)} \trianglerighteq G^{(k+1)} \trianglerighteq \dots$$

Definition

G is solvable if there exists k such that $G^{(k)} = \{1\}$. The minimal k is the derived length of G, $\ell_{der}(G)$, and the group G is called k-step solvable. A solvable group of derived length ≤ 2 is called metabelian.

Solvable groups: immediate properties

Below, no group is assumed to be finitely generated.

Proposition

1. *Every subgroup H of a solvable group G is solvable and* $\ell_{\mathrm{der}}(H) \leqslant \ell_{\mathrm{der}}(G)$.
2. *If G is solvable and $N \lhd G$, then G/N is solvable and* $\ell_{\mathrm{der}}(G/N) \leqslant \ell_{\mathrm{der}}(G)$.
3. *If $N \lhd G$ and both N and G/N are solvable, then G is solvable. Moreover:*
$$\ell_{\mathrm{der}}(G) \leqslant \ell_{\mathrm{der}}(N) + \ell_{\mathrm{der}}(G/N).$$
4. *If G and H are solvable groups then $G \wr H$ is solvable and*
$$\ell_{\mathrm{der}}(G \wr H) \leqslant \ell_{\mathrm{der}}(G) + \ell_{\mathrm{der}}(H).$$

Solvable = poly-abelian

Corollary

A group is solvable if and only if it is poly-abelian.

Proof $\Rightarrow$: The derived series has abelian quotients.
$\Leftarrow$: by induction on the length of the abelian series. If of length one, the group is abelian.
Assume true for length n and let G be poly-abelian with abelian series of length $n+1$.
Let N_1 be the first normal subgroup $\neq G$ in the series.
N_1 poly-abelian with abelian series of length n, hence solvable.
G/N_1 abelian, hence solvable.
We conclude G solvable. □

Corollary

A polycyclic group is solvable.

Examples of solvable groups

Examples

1. *The subgroup $\mathcal{T}_n(\mathbb{K})$ of upper-triangular matrices in $GL(n, \mathbb{K})$, where $\mathbb{K}$ is a field, is solvable.*

   For the next examples, we introduce some terminology: a finite sequence of vector subspaces

   $$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k$$

   in a vector space V is called a flag in V. If the number of subspaces in such a sequence is maximal possible (equal $\dim(V) + 1$), the flag is called full or complete. In other words, $\dim(V_i) = i$ for all subspaces of this sequence.

2. *For a finite-dimensional vector space V, the subgroup G of $GL(V)$ composed of elements g preserving a complete flag in V (i.e. $gV_i = V_i$, for every $g \in G$ and every i) is solvable.*

Comparison between solvable and polycyclic

We now proceed to compare the class of solvable groups with the smaller class of polycyclic groups. In order to do this, we need the following notion.

Definition

A group is said to be noetherian, or to satisfy the maximal condition if for every increasing sequence of subgroups

$$H_1 \leqslant H_2 \leqslant \cdots \leqslant H_n \leqslant \cdots \tag{1}$$

there exists N such that $H_n = H_N$ for every $n \geqslant N$.

Proposition

A group G is noetherian if and only if every subgroup of G is finitely generated.

Proof of characterization of noetherian

Proof $\Rightarrow$ Assume there exists $H \leqslant G$ which is not finitely generated. Pick $h_1 = H \setminus \{1\}$ and let $H_1 = \langle h_1 \rangle$. Inductively, assume that

$$H_1 < H_2 < \dots < H_n$$

is a strictly increasing sequence of finitely generated subgroups of H, pick $h_{n+1} \in H \setminus H_n$, and set $H_{n+1} = \langle H_n, h_{n+1} \rangle$.

We thus have a strictly increasing infinite sequence of subgroups of G, contradicting the assumption that G is noetherian.

$\Leftarrow$ Assume that all subgroups of G are finitely generated.
Consider an increasing sequence of subgroups as in (1). Then $H = \bigcup_{n \geqslant 1} H_n$ is a subgroup, hence generated by a finite set S. There exists N such that $S \subseteq H_N$, hence $H_N = H = H_n$ for every $n \geqslant N$. $\square$

Back to the comparison between solvable and polycyclic

Proposition

A solvable group is polycyclic if and only if it is noetherian.

Proof The 'only if' part follows immediately from the fact that every polycyclic group is solvable, and its subgroups are polycyclic hence finitely generated.

To prove the 'if' part, let G be a noetherian solvable group.

We prove by induction on the derived length k that G is polycyclic.

For $k = 1$ the group is abelian, and since, being noetherian, G is finitely generated, it is polycyclic.

Comparison between solvable and polycyclic, continued

Assume the statement is true for k, consider a solvable group G of derived length $k+1$.
The commutator subgroup $G' \leqslant G$ is also noetherian and solvable of derived length k.
By the induction hypothesis, G' is polycyclic.
The abelianization $G_{ab} = G/G'$ is finitely generated (because G is), hence it is polycyclic.
It follows that G is polycyclic.

Remarks

1. *There are noetherian groups that are not virtually polycyclic, e.g. Tarski monsters: finitely generated groups such that every proper subgroup is cyclic, constructed by Al. Olshanskii.*
2. *Polycyclic groups are noetherian $\Rightarrow$ given any property $(*)$ satisfied by the trivial group $\{1\}$, a polycyclic group contains a maximal subgroup with property $(*)$.*

Noetherian induction for polycyclic groups

We introduce a third type of inductive argument for polycyclic groups: the noetherian induction.

Assume that we have to prove that every polycyclic group has a certain property P. It suffices to check that:

- the trivial group $\{1\}$ has property P (initial case);
- a group G such that all its proper quotients G/N have P must have property P (inductive step).

Indeed, assume that, once all the above was checked, one finds a group G that does not have property P.

Let $(*)$ be the property "K is a normal subgroup such that G/K does not have property P", and let N be a maximal subgroup satisfying $(*)$.

Then G/N is polycyclic, without property P, such that all its proper quotients have property P, contradicting the inductive step.

The Noetherian induction works for any class of Noetherian groups stable by taking quotients.

Example of f.g. solvable non-polycyclic group

Example

Recall that the lamplighter group is the wreath product $G = \mathbb{Z}_2 \wr \mathbb{Z}$, and that it is finitely generated (Ex. Sheet 1).

The commutator subgroup G' coincides with the following subgroup of $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2$:

$$C = \{ f : \mathbb{Z} \to \mathbb{Z}_2 \mid \mathrm{Supp}(f) \text{ has even cardinality} \}, \tag{2}$$

where $\mathrm{Supp}(f) = \{ n \in \mathbb{Z} \mid f(n) = 1 \}$.

[NB. The notation here is additive, the identity element is 0.]

In particular, G' is not finitely generated.

The group G is metabelian (since G' abelian).

The lamplighter group continued

- Not all the subgroups in the lamplighter group G are finitely generated: G' is not, $\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}_2$ is not.
- G is not virtually torsion-free: For any finite-index subgroup $H \leqslant G$, $H \cap \bigoplus_{n\in\mathbb{Z}}\mathbb{Z}_2$ has finite index in $\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}_2$; in particular this intersection is infinite and contains elements of order 2.
- G is not finitely presented.

  The last three statements imply that the lamplighter group is not polycyclic.