C3.3 Differentiable manifolds – Class 3 (27 Nov 2024)

Part B.

6. Let $\alpha \in \Lambda^k V$, $\dim V = n$. Let $A_\alpha : \Lambda^{n-k} V \longrightarrow \Lambda^n V$ be defined by $A_\alpha(\beta) = \alpha \wedge \beta$.

(i) Show that if $\alpha \neq 0$, then $A_\alpha \neq 0$.

proof. Choose a basis $v_1, \dots, v_n$ for V and suppose w.l.o.g. that the component c of $\alpha \neq 0$ in $v_1 \wedge \dots \wedge v_k$ is nonzero. Let

$\beta = v_{k+1} \wedge \dots \wedge v_n$; then $\alpha \wedge \beta = c\, v_1 \wedge \dots \wedge v_n \neq 0$ since every other term in α has one of the vectors $v_{k+1}, \dots, v_n$ and consequently $(\alpha - c\, v_1 \wedge \dots \wedge v_k) \wedge \beta = 0$.

So $A_\alpha(\beta) \neq 0$, $A_\alpha \neq 0$. □

(ii) Show that $\alpha \longmapsto A_\alpha$ is an isomorphism $\Lambda^k V \cong \mathrm{Hom}(\Lambda^{n-k} V, \Lambda^n V)$.

proof. We know that $\dim \Lambda^{n-k} V = \dim \Lambda^k V = \binom{n}{k}$ and $\dim \Lambda^n V = 1$, so $\dim \mathrm{Hom}(\Lambda^{n-k} V, \Lambda^n V) = \binom{n}{k}$.

We showed in (a) that this map is injective, which is sufficient. □

7. Is $S^2 \times \mathbb{RP}^2$ orientable? What about $\mathbb{RP}^2 \times \mathbb{RP}^2$?

Answer: no for both. For $S^2 \times \mathbb{RP}^2$, suppose it is, and pull back the volume form to $S^2 \times S^2$, and consider the antipodal map on the second factor.

For $\mathbb{RP}^2 \times \mathbb{RP}^2$, pull back the volume form to get a volume form on $S^2 \times \mathbb{RP}^2$, which is a contradiction.

These are special cases of a much more general fact:

<u>Proposition.</u> Let M, N be smooth mflds. Then $M \times N$ is orientable if and only if both M and N are orientable.

<u>proof</u>. The "if" direction is problem 5 of part A. For "only if", suppose that $M \times N$ is orientable. Let $U \subseteq N$ be a subset diffeomorphic to $\mathbb{R}^n$. Then $M \times U \subset M \times N$ is an open submanifold and thus is orientable.

Given any atlas $\{V_i, \varphi_i\}$ on M, $\{N_i \times \cancel{U}\, \mathbb{R}^n\}$ is an atlas on $M \times U$, with chart maps $\varphi_i \times \psi$, $\psi : \mathbb{R}^n \xrightarrow{\cong} U$.

Then, given that $(\varphi_i \times \psi)^{-1} \circ (\varphi_j \times \psi)$

$$= (\varphi_i^{-1} \circ \varphi_j) \times \mathrm{id}_{\mathbb{R}^n}$$ has determinant > 0, clearly $\varphi_i^{-1} \circ \varphi_j$ also has determinant > 0 and consequently M is orientable. Similarly for N. $\square$

(The result also follows from some properties of de Rham cohomology such as the Künneth formula.)

8. A <u>Riemann surface</u> is a 2-manifold with an atlas whose transition maps $\mathbb{C} \longrightarrow \mathbb{C}$ are holomorphic. Show that a Riemann surface is orientable.

proof. Let ϕ_i, ϕ_j be ~~charts~~ charts, and let

$f = \phi_j^{-1} \circ \phi_i : \mathbb{C} \to \mathbb{C}$ be the transition map

$f(x,y) = u(x,y) + i v(x,y)$. f is holomorphic, so the
$= f(x+iy) = f(z)$ $= \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$

Cauchy-Riemann equations give $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Then $\det \operatorname{Jac} f = \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \geq 0.$$ f is a local homeo.

so in fact $\det \operatorname{Jac} f > 0$. Thus all transition maps are orientation-preserving and consequently the mfld is orientable.

9. Using the facts that $H^0(X) = \mathbb{R}^{\#\text{ of connected components of } X}$,

$H^0(\mathbb{R}^n) = \mathbb{R}$, $H^{k>0}(\mathbb{R}^n) = 0$ and $H^k(X \times \mathbb{R}^n) \cong H^k(X)$

for all $k, n \geq 0$ and X, show that $H^k(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise,} \end{cases}$

with $H^k(S^0) = \begin{cases} \mathbb{R}^2 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$

Let $U = S^n \setminus \{(1,0,\dots,0)\}$, $V = S^n \setminus \{(-1,0,\dots,0)\}$ and $W = U \cap V$. Then $U \cup V = S^n$ and there are diffeomorphisms

$U \cong \mathbb{R}^n$, $V \cong \mathbb{R}^n$, $W \cong S^{n-1} \times \mathbb{R}$.

Let $B \subseteq A \subseteq S^n$ be open, and write

$\rho_{AB} : \Omega^k(A) \longrightarrow \Omega^k(B)$ for the restriction.

Then there is the exact sequence

$$0 \longrightarrow \Omega^k(S^n) \xrightarrow{\rho_{S^nU} \oplus \rho_{S^nV}} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\rho_{UW} \oplus -\rho_{VW}} \Omega^k(W) \longrightarrow 0.$$

(a) Suppose that $\alpha \in \Omega^k(S^n)$ for $k > 1$ with $d\alpha = 0$. Show that there are $\beta \in \Omega^{k-1}(U)$, $\gamma \in \Omega^{k-1}(V)$ with $\alpha|_U = d\beta$, $\alpha|_V = d\gamma$.

proof. U and V have vanishing ~~...~~ k-th cohomology, $k > 1$, so closed forms are exact.

Let $\delta = \beta|_W - \gamma|_W \in \Omega^{k-1}(W)$. Show that $d\delta = 0$.

proof. $d\delta = d\beta|_W - d\gamma|_W = d\alpha|_W - d\alpha|_W = 0.$

Show that $[\delta] \in H^{k-1}(W)$ depends only on $[\alpha] \in H^k(S^n)$.

proof. First, fix α and let $\beta' = \beta + d\tilde\beta$, $\gamma' = \gamma + d\tilde\gamma$ be other representatives of $[\beta]$ and $[\gamma]$ respectively. Then

$$\delta' - \delta = \beta'|_W - \gamma'|_W - (\beta|_W - \gamma|_W) = d\tilde\beta|_W - d\tilde\gamma|_W$$

$$= d((\tilde\beta - \tilde\gamma)|_W), \text{ so that } [\delta'] = [\delta] \in H^{k-1}(W).$$

So, for α fixed, $[\delta]$ is independent of the choices of β, γ within H^{k-1}.

Now if we choose a different $\alpha' \in [\alpha]$, a priori the choices of $[\beta]$, $[\gamma]$ would have to be different, so we need to prove this separately. In fact, let $\alpha' = \alpha + d\tilde{\alpha}$;

we had ~~$d\beta|_u$~~ $\alpha|_u = d\beta$ and $\alpha|_v = d\gamma$; clearly now if we choose $\beta' = \beta + \tilde{\alpha}|_u$ and $\gamma' = \gamma + \tilde{\alpha}|_v$, we'd have $d\beta' = \alpha'|_u$ and $d\gamma' = \alpha'|_v$, as required.

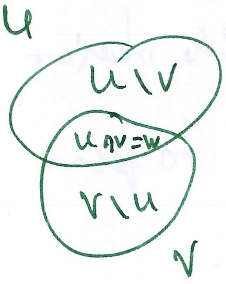
Then
$$\delta' = \beta'|_w - \gamma'|_w = \beta|_w - \gamma|_w + \tilde{\alpha}|_w - \tilde{\alpha}|_w$$
$$= \delta \quad \text{as required.}$$

(b) We thus have a linear map
$$\Phi : H^k(S^n) \longrightarrow H^{k-1}(W), \quad [\alpha] \longmapsto [\delta].$$
Suppose that $[\delta] = \Phi([\alpha]) = 0$. Then $\delta = d\varepsilon$ for some $\varepsilon \in \Omega^{k-2}(W)$. Show that $[\alpha] = 0$, so that Φ is injective.

proof. We follow the hint & let $\{\eta_u, \eta_v\}$ be a partition of unity. First, we compute

$$\beta|_w - d(\eta_v \varepsilon) = \beta|_w - d((1-\eta_u)\varepsilon)$$
$$= \beta|_w - d\varepsilon + d(\eta_u \varepsilon)$$
$$= \gamma|_w + d(\eta_u \varepsilon) \quad \text{since } \delta = d\varepsilon = \beta|_w - \gamma|_w.$$

We know that $\alpha|_u = d\beta$ and $\alpha|_v = d\gamma$. We want to construct a smooth form $\eta \in \Omega^{k-1}(S^n)$ s.t. $d\eta = \alpha$.



Clearly we can define $\eta=\beta$ on $U\setminus V$ and $\eta=\gamma$ on $V\setminus U$. What about on the intersection? Smoothness is the problem. We let

$$\eta = \begin{cases} \beta & \text{on } S^n\setminus V \\ \gamma & \text{on } S^n\setminus U \\ \beta|_W - d(\eta_V\varepsilon) = \gamma|_W + d(\eta_U\varepsilon) & \text{on } W. \end{cases}$$

Then $d\eta$ is just "α" but the additional terms preserve smoothness. Then $d\eta=\alpha$, so $[\alpha]=0\in H^k(S^n)$. □

(c) Suppose that $\delta\in\Omega^{k-1}(W)$ with $d\delta=0$. Show that we can choose α,β,γ s.t. $\Phi([\alpha])=[\delta]$, so that Φ is surjective.

proof. The issue here is that we need both that

$$\beta|_W-\gamma|_W=\delta$$

and that $d\beta, d\gamma$ are both nonvanishing. Treating "δ as a constant form", we see that it suffices to "split" δ via $\beta=\eta_V\delta$ and $\gamma=-\eta_U\delta$. This way, β is defined on all of U, as it vanishes outside of V; similarly for γ. Then

$$\beta|_W-\gamma|_W=(\eta_V+\eta_U)|_W\,\delta=\delta$$

as required, and

$$d\beta=d\eta_V\wedge\delta=d(1-\eta_U)\wedge\delta=-d\eta_U\wedge\delta=d\gamma \text{ on } W;$$

so just set $\alpha=\underbrace{d\beta}_{=d\gamma}$; it extends since β is supported in W. □

(d). Show that $H^k(S^n) \cong H^{k-1}(S^{n-1})$ if $k>1$.

~~proof~~. $W \cong S^{n-1}$, so this is obvious. $\square$

(e) What goes wrong when $k=1$ & how to fix this to show that $H^1(S^1) \cong \mathbb{R}$ & $H^1(S^n) = 0$ for $n>1$?

~~proof~~. What goes wrong here is that $H^{k-1}(\mathbb{R}^n) = H^0(\mathbb{R}^n)$

Consequently, when we showed that $\phi([\alpha])$ $\neq 0$. did not depend on the choice of β, γ, now we can choose different cohomology classes.

There are other ways to fix this, but we can just compute explicitly.

First, note that $H^1(S^1) = \mathbb{R}$. To see this, let $d\theta$ be the coordinate; then if there is a function

$f: S^1 \to \mathbb{R}$ s.t. $df = d\theta$, we must have

$\int_0^{2\pi} df = \int_0^{2\pi} d\theta = 2\pi = f(2\pi) - f(0)$, which is impossible as f is continuous. Consequently $[d\theta] \neq 0$ $\in H^1(S^1)$.

~~Consider any 1-form $h(\theta)\,d\theta$, ... clearly the space H^1 is 1-dimensional, which can be seen by changing coordinates~~

Clearly, as the manifold is 1-dimensional, so is $H^1(S^1)$.

Now we compute $H^1(S^k)$ for $k>1$.

Consider U, V as before; they are topologically $\mathbb{R}^k$ and therefore, given a 1-form ω on S^k, $\omega|_U = df$, $\omega|_V = dg$ for some f, g functions on U, V respectively.

But on W, $d(f-g)|_W = 0 \Rightarrow f-g = \text{constant} = C$; so we just define

$$h = \begin{cases} f & \text{on } U \\ f - C & \text{on } V \end{cases}$$

and then $dh = \omega$;

so that $H^1(S^k) = 0$.

To summarise, we know that:

$$H^k(S^n) = H^{k-1}(S^{n-1}) \quad \text{for all } k>1$$

$$H^1(S^1) \cong \mathbb{R}, \quad H^1(S^k) = 0 \quad \text{for all } k>1.$$

From this, it follows immediately that $H^k(S^k) \cong \mathbb{R}$ for all $k>0$ and that $H^k(S^n) = 0$ if $k \neq 0, n$.

Note that $H^k(S^n) = 0$ if $k>n$ since an n-manifold has no nontrivial higher-order forms.

Fixed solution for problem 7.

<u>Proposition.</u> $M \times N$ is orientable if and only if M and N are both orientable.

<u>proof.</u> Let ω be a volume form on $M \times N$. Let $q \in N$, and let $e^1, \dots, e^n \in T_q N$. Define the m-form η on M by

$$\eta(X_1, \dots, X_m) = \omega(X_1, \dots, X_m, e^1, \dots, e^n).$$

Since ω is a volume form, η is nonvanishing on all of M and consequently is a volume form.

More specifically, in coordinates, if

$$\omega = f(x_1, \dots, x_m, y_1, \dots, y_n)\, dx_1 \wedge \dots dx_m \wedge dy_1 \wedge \dots dy_n,$$

then $\eta = f(x_1, \dots, x_m, q_1, \dots, q_n)\, dx_1 \wedge \dots \wedge dx_m$,

where $(q_1, \dots, q_n)$ are the N-coordinates of q.

Therefore M is orientable. Similarly, N is orientable.