Infinite Groups

Cornelia Druțu

University of Oxford

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Nilpotent and unipotent.

$h \in End(V)$ is nilpotent if $h^k = 0$ for some $k > 0$
$\Leftrightarrow$ in some basis, h can be written as an upper triangular matrix with zeroes on the diagonal.
$\Leftrightarrow$ the only eigenvalue of h is 0.

$g \in End(V)$ is unipotent if $g = \mathrm{id} + h$, where h is nilpotent
$\Leftrightarrow$ the only eigenvalue of g is 1.
$g \in End(V)$ is quasi-unipotent if g^k is unipotent for some $k > 0$
$\Leftrightarrow$ all the eigenvalues of g are roots of unity.

A subgroup $G < GL(V)$ is unipotent (respectively quasi-unipotent) if every element of G is unipotent (respectively, quasi-unipotent).

Kolchin's theorem

Theorem (Kolchin's theorem)

Suppose that $\mathbb{K} = \bar{\mathbb{K}}$ and $G < GL(V)$ is a unipotent subgroup. Then, for an appropriate choice of basis, G is isomorphic to a subgroup of the group of invertible upper-triangular matrices $\mathcal{T}_n(\mathbb{K})$. In particular G is nilpotent.

Proof. The conclusion is equivalent to the statement that G preserves a full flag

$$0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V,$$

where $i = \dim(V_i)$ for each i.
The proof is by induction on the dimension n of V.
Clear for $n = 1$.
We assume that $n > 1$ and that the statement is true for dimensions $< n$.

Proof of Kolchin's theorem continued

Suppose first that the action of G on V is reducible: G preserves a proper subspace $V' \subset V$.
We obtain two induced actions of G on V' (by restriction) and on $V'' = V/V'$ (by projection).
Both actions preserve full flags in V', V'' (induction assumption), and the combination of these flags yields a full G-invariant flag in V.
Assume now that the action of G on V is irreducible.
For $g \in G$ arbitrary the endomorphism $g' = g - I$ is nilpotent, hence it has zero trace.
Therefore, for every $x \in G$, we have

$$tr(g'x) = tr(gx - x) = tr(I) - tr(I) = 0.$$

By Burnside's theorem, G spans $End(V) \Rightarrow$ for each $x \in End(V)$ and each $g \in G$, $tr(g'x) = 0$.
$\tau : End(V) \times End(V) \to \mathbb{K}$ is nondegenerate $\Rightarrow g' = 0$ for all $g \in G$, i.e. $G = \{1\}$. □

Variation of Kolchin's theorem

Theorem (Variation of Kolchin's theorem)

Suppose $\mathbb{K} = \bar{\mathbb{K}}$, $G < GL(V)$ quasiunipotent and, moreover, there exists an upper bound α on the orders of eigenvalues of elements $g \in G$. Then G contains a finite index subgroup isomorphic to a subgroup of the group $\mathcal{U}_n(\mathbb{K})$ of upper triangular matrices with 1 on the diagonal. The index depends only on V and on α. In particular G is virtually nilpotent.

Proof. By induction on the dimension of V.

As before, it suffices to consider the case when G acts irreducibly on V. The orders of the eigenvalues of elements of G are uniformly bounded $\Rightarrow$ the set of traces of elements of G is a certain finite set $C \subset \mathbb{K}$ of cardinality $q = q(\alpha)$.

Our first application to Burnside $\Rightarrow$ G is finite, of cardinality at most q^{n^2}, where $n = \dim V$. □

Solvable linear groups

Theorem (Lie-Kolchin-Mal'cev Theorem)

Let $G \leqslant GL(V)$ be solvable linear, with $n = \dim V$ (G solvable linear of degree n). Then G has a triangularizable normal subgroup K of finite index at most $\mu(n)$, a number that depends only on n.

Definition

Let $\mathcal{X}$ and $\mathcal{Y}$ be two classes of groups.
A group G is $\mathcal{X}$-by-$\mathcal{Y}$ if there exists a short exact sequence

$$\{1\} \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow \{1\},$$

such that $N \in \mathcal{X}$ and $Q \in \mathcal{Y}$.

Solvable linear groups versus nilpotent and polycyclic

Corollary

Let G be a solvable linear group of degree n.

(i) *G has an unipotent-by-abelian normal subgroup of index $\leq \mu(n)$,*

(ii) *(the Zassenhaus Theorem) the derived length of G is at most $\beta(n) := n + \log_2 \mu(n)$.*

This can be combined with the following general result.

Theorem

Every unipotent subgroup of $GL(n, \mathbb{Z})$ is finitely generated.

The two previous results imply the following.

Corollary

Every finitely generated solvable group linear over $\mathbb{Z}$ is polycyclic.

Another comparison between solvable and polycyclic

The converse is also true.

Theorem (Auslander's Theorem)

Every polycyclic group is linear over $\mathbb{Z}$.

Corollary

Every polycyclic group is virtually (finitely generated nilpotent)-by-(f.g. abelian).

We have thus obtained the following way of distinguishing polycyclic groups among solvable groups.

Theorem

Given a f.g. solvable group G, the following are equivalent:

- *G is polycyclic;*
- *G is linear over $\mathbb{Z}$.*

Comparison between solvable and nilpotent: growth

Final topic of this course: distinguishing f.g nilpotent groups in the larger class of f.g. solvable groups *via* their growth.

This is the celebrated Milnor-Wolf Theorem.

Byproducts of the proof: new features that allow to distinguish between solvable and polycyclic, polycyclic and nilpotent.

Let $G = \langle S \rangle$, where S finite, $S^{-1} = S$, $1 \notin S$.
Let dist_S be the word metric associated to S.
The growth function of G with respect to S is

$$\mathfrak{G}_{G,S}(R) := \mathrm{card}\, \bar{B}(1, R).$$

Question: How much does $\mathfrak{G}_{G,S}$ depend on S?

Growth functions

Definition

Given $f, g : X \to \mathbb{R}$ with $X \subset \mathbb{R}$, we define an asymptotic inequality $f \preceq g \Leftrightarrow \exists a, b > 0,\ c \geqslant 0$ and $x_0 \in \mathbb{R}$ such that $\forall x \in X, x \geqslant x_0$, $bx + c \in X$ and $f(x) \leqslant ag(bx + c)$.

$f \asymp g \Leftrightarrow f \preceq g$ and $g \preceq f$; we say that f and g are asymptotically equal.

Lemma

Assume that (G, dist_S) and (H, dist_X) are bi-Lipschitz equivalent, i.e. $\exists L > 0$ and a bijection $f : G \to H$ such that

$$\frac{1}{L}\mathrm{dist}_S(g, g') \leqslant \mathrm{dist}_X(f(g), f(g')) \leqslant L\mathrm{dist}_S(g, g')\,, \forall g, g' \in G\,. \quad (1)$$

Then $\mathfrak{G}_{G,S} \asymp \mathfrak{G}_{H,X}$.
In particular true when $(H, \mathrm{dist}_X) = (G, \mathrm{dist}_{S'})$, $G = \langle S' \rangle$.

Growth functions

Corollary

If S, S' are two finite generating sets of G then $\mathfrak{G}_S \asymp \mathfrak{G}_{S'}$. Thus, one can speak of growth function $\mathfrak{G}_G$ of a group G, well defined up to $\asymp$.

Examples

1. If $G = \mathbb{Z}^k$ then $\mathfrak{G}_S \asymp x^k$ for every finite generating set $S = S^{-1}$.
2. If $G = F_k$, the free group of finite rank $k \geqslant 2$, and X is the set of k letters/symbols then

$$\mathfrak{G}_{X \sqcup X^{-1}}(n) = 1 + (q^n - 1)\frac{q+1}{q-1}, \quad q = 2k - 1.$$