Axiomatic Set Theory: Problem sheet 4

A.

1. Prove 7.1.2, 7.1.3, and 7.1.6.

2. Prove 7.1.11 (30), ie. that "x is a finite sequence of elements of y" (ie. $x \in {}^{<\omega}y$) is Σ_0^{ZF}, assuming that (1)–(29) of 7.11 are all Σ_0^{ZF}.

B.

3. Prove that "x is a well-ordering of y" is Δ_1^{ZF}.

4. Show that for every Σ_1 formula $\phi(x_1, \ldots, x_n)$, there exists a corresponding Σ_0 formula $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that

$$ZF \vdash \forall x_1, \ldots x_n(\phi(x_1, \ldots, x_n) \leftrightarrow \exists y_1, \ldots, y_m \psi(x_1, \ldots, x_n, y_1, \ldots, y_m)).$$

5. Prove that ordinal addition, multiplication and exponentiation are Δ_1^{ZF}.

6. Prove that for any infinite cardinal κ, $\mathrm{cof}(\kappa)$ is a regular cardinal.

7. Suppose κ, λ are infinite cardinals such that $\kappa \geq \lambda$. Prove that if $\lambda \geq \mathrm{cof}(\kappa)$, then $\kappa^\lambda > \kappa$. Suppose now that $\lambda < \mathrm{cof}(\kappa)$, and that κ has the property that for any cardinal μ, if $\mu < \kappa$ then $2^\mu \leq \kappa$. Prove that $\kappa^\lambda = \kappa$. Hence show that if GCH is assumed, then for any infinite cardinals κ, λ with $\kappa \geq \lambda$, we have $\kappa^\lambda = \kappa$ or κ^+.

C.

8. Suppose κ is an *uncountable regular* cardinal. Let $g : \kappa \to \kappa$ be any function. Prove that for any $\alpha < \kappa$, there exists $\beta < \kappa$, with $\alpha \leq \beta$, such that β is closed under g (ie. for all $\gamma < \beta$, $g(\gamma) < \beta$).

9. Let κ be an uncountable regular cardinal with the property that for any cardinal $\mu < \kappa$, we have $2^\mu < \kappa \ldots$ (*).

Prove that (i) if α is any cardinal and $\alpha < \kappa$, then $|V_\alpha| < \kappa$, (ii) $|V_\kappa| = \kappa$, (iii) if κ is regular, then $\langle V_\kappa, \in\rangle \vDash$ ZFC.

(For (iii) you need consider only the replacement scheme, since we essentially showed that if α is a limit ordinal and $\alpha > \omega$, then $\langle V_\alpha, \in\rangle$ satisfies all the axioms of ZFC except, possibly, replacement.)

Deduce that in ZFC one cannot prove the existence of a cardinal that satisfies (*).