B2.2 Commutative Algebra

Sheet 1 — HT25

Sections 1-5

Section A

1. Let R be a ring. Show that the Jacobson radical of R coincides with the set

$$\{x \in R \mid 1 - xy \text{ is a unit for all } y \in R\}.$$

Dawid Kielak: kielak@maths.ox.ac.uk

Section B

2. Let R be a ring.
   (a) Show that if $P(x) = a_0 + a_1 x + \cdots + a_k x^k \in R[x]$ is a unit of $R[x]$ then a_0 is a unit of R and a_i is nilpotent for all $i \geqslant 1$.
   (b) Show that the Jacobson radical and the nilradical of $R[x]$ coincide.

3. Let R be a ring and let $N \subseteq R$ be its nilradical. Show that the following are equivalent:
   (a) R has exactly one prime ideal.
   (b) Every element of R is either a unit or is nilpotent.
   (c) R/N is a field.

4. Let R be a ring and let $I \subseteq R$ be an ideal. Let $S = \{1 + r \mid r \in I\}$.
   (a) Show that S is a multiplicative set.
   (b) Show that the ideal generated by the image of I in R_S is contained in the Jacobson radical of R_S.
   (c) Prove the following generalisation of Nakayama's lemma:

   Lemma. *Let M be a finitely generated R-module and suppose that $IM = M$. Then there exists $r \in R$, such that $r - 1 \in I$ and $rM = 0$.*

5. Let R be a ring and let M be a finitely generated R-module. Let $\phi\colon M \to M$ be a surjective homomorphism of R-modules. Prove that ϕ is injective, and is thus an automorphism. [Hint: use ϕ to construct a structure of $R[x]$-module on M and use the previous question.]

6. Let R be a ring. Let $\mathcal{S}$ be the subset of the set of ideals of R defined as follows: an ideal I is in $\mathcal{S}$ if and only if all the elements of I are zero-divisors. Show that $\mathcal{S}$ has maximal elements (for the relation of inclusion) and that every maximal element is a prime ideal. Show that the set of zero-divisors of R is a union of prime ideals.

Dawid Kielak: kielak@maths.ox.ac.uk

Section C

7. Let R be a ring. Consider the inclusion relation on the set $\mathrm{Spec}(R)$. Show that there are minimal elements in $\mathrm{Spec}(R)$.

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