B2.2 Commutative Algebra

Sheet 2 — HT25

Sections 1-8

Section A

1. Consider the ideals $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$ and $\mathfrak{m} = (x, y, z)$ of $K[x, y, z]$, where K is a field. Show that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of $\mathfrak{p}_1 \cdot \mathfrak{p}_2$. Determine the isolated and the embedded prime ideals of $\mathfrak{p}_1 \cdot \mathfrak{p}_2$.

   Solution: For future reference, note that we have

   $$\mathfrak{m}^2 = ((x) + (y) + (z))^2 = (x^2, y^2, z^2, xy, xz, yz)$$

   and

   $$\mathfrak{p}_1 \cdot \mathfrak{p}_2 = ((x) + (y))((x) + (z)) = (x^2, xz, yx, yz).$$

   We have $\mathfrak{p}_1 \cdot \mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$ and we also clearly have $\mathfrak{p}_1 \cdot \mathfrak{p}_2 \subseteq \mathfrak{m}^2$ since $\mathfrak{p}_1, \mathfrak{p}_2 \subseteq \mathfrak{m}$. Thus we have $\mathfrak{p}_1 \cdot \mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Note that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime since the rings $K[x, y, z]/\mathfrak{p}_1 \simeq K[z]$ and $K[x, y, z]/\mathfrak{p}_2 \simeq K[y]$ are domains. Note also that $\mathfrak{m}$ is a maximal ideal, since $K[x, y, z]/\mathfrak{m} \simeq K$ is a field. Thus $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{m}^2$ are primary (see after Lemma 6.4 for the latter). The radicals of the ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{m}^2$ are $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{m}$ (see again Lemma 6.4 for the latter). These three ideals are distinct. Finally, we have $\mathfrak{p}_1 \not\supseteq \mathfrak{p}_2 \cap \mathfrak{m}^2$ (because $z^2 \notin \mathfrak{p}_1$ but $z^2 \in \mathfrak{p}_2 \cap \mathfrak{m}^2$), $\mathfrak{p}_2 \not\supseteq \mathfrak{p}_1 \cap \mathfrak{m}^2$ (because $y^2 \notin \mathfrak{p}_2$ but $y^2 \in \mathfrak{p}_1 \cap \mathfrak{m}^2$) and $\mathfrak{m}^2 \not\supseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$ (because $x \notin \mathfrak{m}^2$ but $x \in \mathfrak{p}_2 \cap \mathfrak{p}_2$). Hence if $\mathfrak{p}_1 \cdot \mathfrak{p}_2 = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ then this decomposition is indeed primary and minimal. Thus we only have to show that $\mathfrak{p}_1 \cdot \mathfrak{p}_2 \supseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$.

   From the above, we have to show that

   $$(x, y) \cap (x, z) \cap (x^2, y^2, z^2, xy, xz, yz) \subseteq (x^2, xz, yx, yz).$$

   This is immediate, since all the ideals we are considering have the property that a polynomial lies in such an ideal if and only if all of the monomial summands of the polynomial lie in the ideal.

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Section B

2. Let K be a field. Show that the ideal $(x^2, xy, y^2) \subseteq K[x,y]$ is a primary ideal, which is not irreducible.

3. Let R be a noetherian ring and let T be a finitely generated R-algebra. Let G be a finite subgroup of the group of automorphisms of T as a R-algebra. Let T^G be the fixed point set of G (ie the subset of T, which is fixed by all the elements of G).

   (a) Show that T is integral over T^G.

   (b) Show that T^G is a subring of T, which contains the image of R and that T^G is finitely generated over R.

4. Show that $\mathbb{Z}$ is integrally closed and that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i)$ is $\mathbb{Z}[i]$.

5. Let S be a ring and let $R \subseteq S$ be a subring of S. Suppose that R is integrally closed in S. Let $P(x) \in R[x]$ and suppose that $P(x) = Q(x)J(x)$, where $Q(x), J(x) \in S[x]$ and $Q(x)$ and $J(x)$ are monic. Show that $Q(x), J(x) \in R[x]$. Use this to give a new proof of the fact that if $T(x) \in \mathbb{Z}[x]$ and $T(x) = T_1(x)T_2(x)$, where $T_1(x), T_2(x) \in \mathbb{Q}[x]$ are monic polynomials, then $T_1(x), T_2(x) \in \mathbb{Z}[x]$.

6. Let R be a subring of a ring T and suppose that T is integral over R. Let $\mathfrak{p}$ be prime ideal of R and let $\mathfrak{q}$ be a prime ideal of T. Suppose that $\mathfrak{q} \cap R = \mathfrak{p}$. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_k$ be primes ideal of R and suppose that $\mathfrak{p}_1 = \mathfrak{p}$. Show that there are prime ideals $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_k$ of T such that $\mathfrak{q}_1 = \mathfrak{q}$ and such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for all $i \in \{1, \ldots, k\}$.

7. Let R be a ring. Let $\mathcal{S}$ be the set of ideals in R that are not finitely generated; assume that $\mathcal{S} \neq \emptyset$.

   (a) Show that $\mathcal{S}$ has at least one maximal element.

   (b) Let I be maximal element of $\mathcal{S}$ (with respect to the relation of inclusion). Show that I is prime.

   (c) Suppose that all the prime ideals of R are finitely generated. Prove that R is noetherian.

   [Hint: exploit the fact that R/I is noetherian.]

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Section C

8. Let R be a ring. Let $\mathcal{S}$ be the set of non-principal ideals in R; assume that $\mathcal{S} \neq \emptyset$. Prove that $\mathcal{S}$ admits maximal elements, and that every such element a prime ideal.

   Solution: The existence of maximal elements follows from Zorn's lemma. Let I be one such. Let $x, y \notin I$ and suppose for contradiction that $xy \in I$. Let $I_x = (x) + I$. By assumption, we have $I_x = (g_x)$ for some $g_x \in R$. Let $\phi\colon R \to I_x$ be the surjection of R-modules given by the formula $\phi(r) = rg_x$. We then have $I \subseteq \phi^{-1}(I)$.

   Suppose first that $I = \phi^{-1}(I)$. In other words, for all $r \in R$, we have $rg_x \in I$ if and only if $r \in I$. This contradicts the fact that $yg_x \in I$. So we conclude that $I \subset \phi^{-1}(I)$. From the definition of I, we then see that $\phi^{-1}(I)$ is a principal ideal of R, and hence so is $I = \phi(\phi^{-1}(I))$. This is a contradiction.

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