B2.2 Commutative Algebra

Sheet 2 — HT25

Sections 1-8

Section A

1. Consider the ideals $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$ and $\mathfrak{m} = (x, y, z)$ of $K[x, y, z]$, where K is a field. Show that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of $\mathfrak{p}_1 \cdot \mathfrak{p}_2$. Determine the isolated and the embedded prime ideals of $\mathfrak{p}_1 \cdot \mathfrak{p}_2$.

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Section B

2. Let K be a field. Show that the ideal $(x^2, xy, y^2) \subseteq K[x, y]$ is a primary ideal, which is not irreducible.

3. Let R be a noetherian ring and let T be a finitely generated R-algebra. Let G be a finite subgroup of the group of automorphisms of T as a R-algebra. Let T^G be the fixed point set of G (ie the subset of T, which is fixed by all the elements of G).
   (a) Show that T is integral over T^G.
   (b) Show that T^G is a subring of T, which contains the image of R and that T^G is finitely generated over R.

4. Show that $\mathbb{Z}$ is integrally closed and that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i)$ is $\mathbb{Z}[i]$.

5. Let S be a ring and let $R \subseteq S$ be a subring of S. Suppose that R is integrally closed in S. Let $P(x) \in R[x]$ and suppose that $P(x) = Q(x)J(x)$, where $Q(x), J(x) \in S[x]$ and $Q(x)$ and $J(x)$ are monic. Show that $Q(x), J(x) \in R[x]$. Use this to give a new proof of the fact that if $T(x) \in \mathbb{Z}[x]$ and $T(x) = T_1(x)T_2(x)$, where $T_1(x), T_2(x) \in \mathbb{Q}[x]$ are monic polynomials, then $T_1(x), T_2(x) \in \mathbb{Z}[x]$.

6. Let R be a subring of a ring T and suppose that T is integral over R. Let $\mathfrak{p}$ be prime ideal of R and let $\mathfrak{q}$ be a prime ideal of T. Suppose that $\mathfrak{q} \cap R = \mathfrak{p}$. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_k$ be primes ideal of R and suppose that $\mathfrak{p}_1 = \mathfrak{p}$. Show that there are prime ideals $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_k$ of T such that $\mathfrak{q}_1 = \mathfrak{q}$ and such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for all $i \in \{1, \ldots, k\}$.

7. Let R be a ring. Let $\mathcal{S}$ be the set of ideals in R that are not finitely generated; assume that $\mathcal{S} \neq \emptyset$.
   (a) Show that $\mathcal{S}$ has at least one maximal element.
   (b) Let I be maximal element of $\mathcal{S}$ (with respect to the relation of inclusion). Show that I is prime.
   (c) Suppose that all the prime ideals of R are finitely generated. Prove that R is noetherian.

   [Hint: exploit the fact that R/I is noetherian.]

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Section C

8. Let R be a ring. Let $\mathcal{S}$ be the set of non-principal ideals in R; assume that $\mathcal{S} \neq \emptyset$. Prove that $\mathcal{S}$ admits maximal elements, and that every such element a prime ideal.

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