B2.2 Commutative Algebra

Sheet 3 — HT25

Sections 1-10

Section A

1. Let R be a subring of a ring T. Suppose that T is integral over R. Let $\mathfrak{p}$ be a prime ideal of R and let $\mathfrak{q}_1, \mathfrak{q}_2$ be prime ideals of T such that $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R = \mathfrak{p}$. Show that if $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ then $\mathfrak{q}_1 = \mathfrak{q}_2$.

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Section B

2. Let R be a ring. Show that the two following conditions are equivalent:

   (a) R is a Jacobson ring.

   (b) If $\mathfrak{p} \in \operatorname{Spec}(R)$ and $R/\mathfrak{p}$ contains an element b such that $(R/\mathfrak{p})[b^{-1}]$ is a field, then $R/\mathfrak{p}$ is a field.

   Here we write $(R/\mathfrak{p})[b^{-1}]$ for the localisation of $R/\mathfrak{p}$ at the multiplicative subset $1, b, b^2, \ldots$.

3. Let k be field and let R be a finitely generated algebra over k. Show that the two following conditions are equivalent:

   (a) $\operatorname{Spec}(R)$ is finite.

   (b) R is finite over k.

4. Let k be an algebraically closed field. Let $P_1, \ldots, P_d \in k[x_1, \ldots, x_d]$. Suppose that the set
$$\{(y_1, \ldots, y_d) \in k^d \mid P_i(y_1, \ldots, y_d) = 0 \,\forall i \in \{1, \ldots, d\}\}$$
   is finite. Show that
$$\operatorname{Spec}(k[x_1, \ldots, x_d]/(P_1, \ldots, P_d))$$
   is finite.

5. Let R be a ring and let R_0 be the prime ring of R (see the preamble of the notes for the definition). Suppose that R is a finitely generated R_0-algebra. Suppose also that R is a field. Prove that R is a finite field.

6. Let k be a field and let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_d]$. Show that there are polynomials $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \ldots, P_d(x_1, \ldots, x_d)$ such that $\mathfrak{m} = (P_1, \ldots, P_d)$.

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Section C

7. Let R be a domain. Show that $R[x]$ is integrally closed if R is integrally closed.

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