B2.2 Commutative Algebra

Sheet 3 — HT25

Sections 1-10

Section A

1. Let R be a subring of a ring T. Suppose that T is integral over R. Let $\mathfrak{p}$ be a prime ideal of R and let $\mathfrak{q}_1, \mathfrak{q}_2$ be prime ideals of T such that $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R = \mathfrak{p}$. Show that if $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ then $\mathfrak{q}_1 = \mathfrak{q}_2$.

   Solution: The ring $R/\mathfrak{p}$ is can be viewed as a subring of $T/\mathfrak{q}_1$ and by assumption we have $(\mathfrak{q}_2 \mod \mathfrak{q}_1) \cap R/\mathfrak{p} = (0)$. We may thus assume without loss of generality that R and T to be domains and that $\mathfrak{q}_1$ and $\mathfrak{p}$ are zero ideals.

   Now let $e \in \mathfrak{q}_2 \setminus \{0\}$ and let $P(x) \in R[x]$ be a non-zero monic polynomial such that $P(e) = 0$. Since T is a domain, we may assume that the constant coefficient of $P(x)$ is non-zero (otherwise, replace $P(x)$ by $P(x)/x^k$ for a suitable $k \geqslant 1$). But then the constant term $P(0)$ is a linear combination of positive powers of e (since $P(e) = 0$), so $P(0) \in R \cap \mathfrak{q}_2 = (0)$. This is a contradiction.

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Section B

2. Let R be a ring. Show that the two following conditions are equivalent:
   (a) R is a Jacobson ring.
   (b) If $\mathfrak{p} \in \mathrm{Spec}(R)$ and $R/\mathfrak{p}$ contains an element b such that $(R/\mathfrak{p})[b^{-1}]$ is a field, then $R/\mathfrak{p}$ is a field.

   Here we write $(R/\mathfrak{p})[b^{-1}]$ for the localisation of $R/\mathfrak{p}$ at the multiplicative subset $1, b, b^2, \ldots$.

3. Let k be field and let R be a finitely generated algebra over k. Show that the two following conditions are equivalent:
   (a) $\mathrm{Spec}(R)$ is finite.
   (b) R is finite over k.

4. Let k be an algebraically closed field. Let $P_1, \ldots, P_d \in k[x_1, \ldots, x_d]$. Suppose that the set
$$\{(y_1, \ldots, y_d) \in k^d \mid P_i(y_1, \ldots, y_d) = 0 \,\forall i \in \{1, \ldots, d\}\}$$
is finite. Show that
$$\mathrm{Spec}(k[x_1, \ldots, x_d]/(P_1, \ldots, P_d))$$
is finite.

5. Let R be a ring and let R_0 be the prime ring of R (see the preamble of the notes for the definition). Suppose that R is a finitely generated R_0-algebra. Suppose also that R is a field. Prove that R is a finite field.

6. Let k be a field and let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_d]$. Show that there are polynomials $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \ldots, P_d(x_1, \ldots, x_d)$ such that $\mathfrak{m} = (P_1, \ldots, P_d)$.

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Section C

7. Let R be a domain. Show that $R[x]$ is integrally closed if R is integrally closed.

   Solution: Suppose that R is integrally closed in its fraction field K. The fraction field of $R[x]$ is $K(x) = (K[x])(K[x] \setminus \{1\})^{-1}$. Let $Q(x) \in K(x)$ and suppose that $Q(x)$ is integral over $R[x]$. Suppose for a contradiction that $Q(x) \notin R[x]$, and take $Q(x)$ of smallest possible degree. Clearly $Q(x)$ is not the zero polynomial.

   Then $Q(x)$ is in particular integral over $K[x]$ and we saw in the solution of Question 4, sheet 2, that $K[x]$ is integrally closed, since it is a PID. So we deduce that $Q(x) \in K[x]$.

   Now let

   $$Q^n + P_{n-1}Q^{n-1} + \cdots + P_0 = 0$$

   be a non trivial integral equation for Q with $P_i \in R[x]$ for all n. Evaluating at $x = 0$ shows that the constant term of $Q(x)$ is integral over R, and hence lies in R. Since the integral closure of $R[x]$ is a ring, we may subtract the constant term, and assume that the constant term of Q is zero. But then we can also divide by a power of x, and decrease the degree of Q. Contradiction.

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