B2.2 Commutative Algebra

Sheet 3 — HT25

Sections 1-10

Section A

1. Let R be a noetherian domain. Let $\mathfrak{m}$ be a maximal ideal in R. Let $r \in R\backslash\{0\}$ and suppose that (r) is a $\mathfrak{m}$-primary ideal. Show that $\text{height}((r)) = 1$.

   Solution: By assumption, the nilradical of (r) is $\mathfrak{m}$. Since the nilradical is the intersection of all the prime ideals containing (r), we see that every prime ideal containing (r) also contains $\mathfrak{m}$. On the other hand, a prime ideal containing $\mathfrak{m}$ must be equal to $\mathfrak{m}$. We conclude that $\mathfrak{m}$ is the only prime ideal containing (r). In particular, $\mathfrak{m}$ is minimal among the prime ideals containing (r) and thus $\text{height}((r)) = \text{height}(\mathfrak{m}) \leqslant 1$ by Krull's principal ideal theorem. On the other hand, $\text{height}(\mathfrak{m}) = 1$, since we have the chain $\mathfrak{m} \supset (0)$ (note that R is a domain).

2. Let R be a PID. Show that $\dim R \leqslant 1$, and that $\dim R = 0$ if and only if R is a field.

   Solution: We have the prime ideal (0), since R is a domain. If R is a field, then we have no other prime ideals, and $\dim R = 0$.

   If R is not a field, then it has at least one non-trivial proper prime ideal. Every such ideal is maximal (see Sheet 0), and hence $\dim R = 1$.

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Section B

3. Let A, B be integral domains and suppose that $A \subseteq B$. Suppose that A is integrally closed and that B is integral over A. Let

$$\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$$

be a descending chain of prime ideals in A. Let $k \in \{0, \ldots, n-1\}$ and let

$$\mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \cdots \supset \mathfrak{q}_k$$

be a descending chain of prime ideals in B, such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \in \{0, \ldots, k\}$. Then there is a descending chain of prime ideals

$$\mathfrak{q}_k \supset \mathfrak{q}_{k+1} \supset \cdots \supset \mathfrak{q}_n$$

such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \in \{k+1, \ldots, n\}$. This is the "Going-down Theorem", see AT, Th. 5.16, p. 64.

Let L (resp. K) be the fraction field of B (resp. A). Prove the Going-down Theorem when L is a (finite) Galois extension of K.

4. Let R be an integrally closed domain. Let $K = \mathrm{Frac}(R)$. Let $L|K$ be an algebraic field extension. Show that an element $e \in L$ is integral over R if and only if the minimal polynomial of e over K has coefficients in R.

5. Let R be a PID. Suppose that $2 = 1 + 1$ is a unit in R. Let $c_1, \ldots, c_t \in R$ be distinct irreducible elements with $t \geqslant 1$, and let $c = c_1 \cdots c_t$. Show that the ring $R[x]/(x^2 - c)$ is a Dedekind domain. Use this to show that $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is a Dedekind domain.

6. Let R be a PID. Let $c_1, c_2 \in R$ be two distinct irreducible elements and let $c = c_1 \cdot c_2$. Consider ideals of $R[x]/(x^2 - c)$. Show that $(c) = (x, c_1)^2 \cdot (x, c_2)^2$ and that the ideals (x, c_i) are prime.

7. Let R be a ring (not necessarily noetherian). Suppose that $\dim(R) < \infty$. Show that $\dim(R[x]) \leqslant 1 + 2\dim(R)$.

8. Let R be a Dedekind domain. Let I be a non zero ideal in R. Show that every ideal in R/I is principal. Deduce that every ideal in a Dedekind domain can be generated by two elements.

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9. Let A (resp. B) be a noetherian local ring with maximal ideal $\mathfrak{m}_A$ (resp. $\mathfrak{m}_B$). Let $\phi : A \to B$ be a ring homomorphism and suppose that $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ (such a homomorphism is said to be 'local').

   Suppose that

   (a) B is finite over A via ϕ;

   (b) the map $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ induced by ϕ is surjective;

   (c) the map $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ induced by ϕ is bijective.

   Prove that ϕ is surjective. [Hint: use Nakayama's lemma twice].

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Section C

10. Let R be a Dedekind domain. Show that R is a PID if and only if it is a UFD.

Solution: Every PID is a UFD.

For the converse, first note that it is enough to prove that all prime ideals are principal, since every non-trivial proper ideal in a Dedekind domain is a product of prime ideals.

Let $\mathfrak{p}$ be a non-trivial prime ideal in R. Since R is a UFD, there is a prime element $p \in \mathfrak{p}$. Hence we have the inclusions

$$(0) \subset (p) \subseteq \mathfrak{p},$$

and since $\dim R = 1$ we must have $\mathfrak{p} = (p)$.

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