C3.8 Analytic Number Theory

Sheet 2 — MT24

Section A

1. Evaluate the sum $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$.

   Solution: We use the fact, established in lectures, that $D\mu(s) = \sum_n \mu(n) n^{-s} = \frac{1}{\zeta(s)}$. Thus the sum in question is $1/\zeta(2)$. But it is well-known that $\zeta(2) = \sum_n n^{-2} = \frac{\pi^2}{6}$, so the answer is $6/\pi^2$.

2. Give a simple description of the function $\phi \star 1$.

   Solution: By definition
   $$\phi \star 1(n) = \sum_{d|n} \phi(\frac{n}{d}).$$
   Now $\phi(n/d)$ is precisely the number of $m \leqslant n$ whose highest common factor with n is precisely d. Thus the sum here is simply counting the number of $m \leqslant n$, or in other words $\phi \star 1(n) = n$.

Ben Green: ben.green@maths.ox.ac.uk

Section B

3. Establish the following Dirichlet series:

   (i) $\sum_n \tau(n)n^{-s} = \zeta(s)^2$ for $\Re s > 1$;

   (ii) $\sum_n \phi(n)n^{-s} = \frac{\zeta(s-1)}{\zeta(s)}$ for $\Re s > 2$;

   (iii) $\sum_n \sigma(n)n^{-s} = \zeta(s)\zeta(s-1)$.

   (iv) If $\lambda(n)$ is the Liouville function, that is to say the unique completely multiplicative function equal to -1 on the primes, then $\sum_n \lambda(n)n^{-s} = \frac{\zeta(2s)}{\zeta(s)}$ for $\Re s > 1$.

   Solution: (i) We have $\tau = 1 \star 1$. Thus from the link between Dirichlet convolution and Dirchlet series we have $D_\tau = (D_1)^2$.

   (ii) We have $\phi \star 1 = \iota$, where $\iota(n) = n$. Noting that $D_\iota(s) = \zeta(s-1)$ when $\Re s > 2$, the result follows.

   (iii) This follows from the fact that, by definition, $\sigma = 1 \star \iota$.

   (iv) This is probably most easily done via Euler products. We have

$$D_\lambda(s) = \prod_p \big(1 - \frac{1}{p^s} + \frac{1}{p^{2s}} - \dots\big) = \prod_p \big(1 + \frac{1}{p^s}\big)^{-1}.$$

   However,

$$1 + \frac{1}{p^s} = \frac{1 - \frac{1}{p^{2s}}}{1 - \frac{1}{p^s}}.$$

   Alternatively, one may note that $\lambda = \mu \star 1_S$, where S is the set of squares, then observe that $D_{1_S}(s) = \zeta(2s)$.

4. Obtain an asymptotic for $\sum_{n \leqslant X} \tau(n)$.

   Solution: We have

$$\sum_{n \leqslant X} \tau(n) = \sum_{ab \leqslant X} 1 = \sum_{a \leqslant X} \Big\lfloor \frac{X}{a} \Big\rfloor = \sum_{a \leqslant X} \frac{X}{a} + O(X) = X \log X + o(X \log X).$$

5. True or false? There is a constant C such that $\tau(n) \leqslant \log^C n$ for all sufficiently large n. Justify your answer.

Ben Green: ben.green@maths.ox.ac.uk

Solution: This is false. Let $p_1 < p_2 < \dots, < p_r$ be the first r primes, and consider $n = (p_1 \cdots p_r)^m$. We have $\tau(n) = (m+1)^r > m^r$. However, $\log n = \gamma m$ where $\gamma = \log(p_1 \dots p_r)$ is independent of m. (One could consider n of the form $n = \prod_{p \leqslant X} p$. Then, as we showed in lectures, $n \leqslant e^{O(X)}$. However, $\tau(n) = 2^{\pi(X)}$, which is vastly bigger than X^C.)

Ben Green: ben.green@maths.ox.ac.uk

6. Show that

$$\sum_n \Lambda(n) \Big\lfloor \frac{Y}{n} \Big\rfloor = \sum_{n \leqslant Y} \log n.$$

By considering $Y = X$ and $Y = X/2$, use this to prove that

$$\sum_{X/2 < n \leqslant X} \Lambda(n) \ll X.$$

Solution: We have $\lfloor \frac{Y}{n} \rfloor = \sum_{x \leqslant Y, n | x} 1$. Therefore

$$\sum_{n \leqslant Y} \Lambda(n) \Big\lfloor \frac{Y}{n} \Big\rfloor = \sum_{n \leqslant X} \sum_{x \leqslant Y : n | x} \Lambda(n).$$

Swapping the order of summation, this is

$$\sum_{x \leqslant Y} \sum_{n : n | x} \Lambda(n) = \sum_{x \leqslant Y} (1 \star \Lambda)(x) = \sum_{x \leqslant Y} \log x,$$

which is the result stated.

We showed on Sheet 1 (Q8) that $\sum_{x \leqslant Y} \log x = Y \log Y - Y + o(Y)$. Thus

$$\sum_n \Lambda(n) \Big\lfloor \frac{X}{n} \Big\rfloor = X \log X - X + o(X)$$

and

$$\sum_n \Lambda(n) \Big\lfloor \frac{X}{2n} \Big\rfloor = \frac{1}{2} X \log\Big(\frac{X}{2}\Big) - \frac{X}{2} + o(X).$$

Subtracting twice the second expression from the first yields

$$\sum_n \Lambda(n) \Big(\Big\lfloor \frac{X}{n} \Big\rfloor - 2 \Big\lfloor \frac{X}{2n} \Big\rfloor \Big) = X \log 2 + o(X).$$

Note, however, that the bracketed expression is 1 when $X/2 \leqslant n < X$. Therefore

$$\sum_{X/2 < n \leqslant X} \Lambda(n) \ll X.$$

Remark. This is very closely related to the proof we gave in lectures that $\pi(X) \ll \frac{X}{\log X}$.

Ben Green: ben.green@maths.ox.ac.uk

7. Write $L(X) := \sum_{n \leqslant X} \lambda(n)$ and $M(X) := \sum_{n \leqslant X} \mu(n)$. Establish the relations

$$L(X) = \sum_{d \leqslant \sqrt{X}} M(X/d^2)$$

and

$$M(X) = \sum_{d \leqslant \sqrt{X}} \mu(d) L(X/d^2),$$

and hence conclude that the statements $L(X) = o(X)$ and $M(X) = o(X)$ are equivalent.

Solution: The first statement is easier. We have $\lambda = 1_S \star \mu$, where 1_S is the characteristic function of the squares, so

$$L(X) = \sum_{n \leqslant X} \lambda(n) = \sum_{n \leqslant X} \sum_{d^2 | n} \mu\Big(\frac{n}{d^2}\Big) = \sum_{d \leqslant \sqrt{X}} \sum_{m \leqslant \frac{X}{d^2}} \mu(m) = \sum_{d \leqslant \sqrt{X}} M(X/d^2).$$

The second statement is a bit trickier and requires us to invert the relation $\lambda = 1_S \star \mu$. The key to this is to note that "Möbius inversion works on the squares" and in fact $\tilde{1}_S \star 1_S = \delta$, where $\tilde{1}_S(n) = \mu(\sqrt{n})$ if n is a square and 0 otherwise. This may be verified directly. It follows that $\mu = \tilde{1}_S \star \lambda$. With this in hand, the argument is essentially the same as before:

$$M(X) = \sum_{n \leqslant X} \mu(n) = \sum_{n \leqslant X} \sum_{d^2 | n} \mu(d) \lambda\Big(\frac{n}{d^2}\Big) = \sum_{d \leqslant \sqrt{X}} \mu(d) \sum_{m \leqslant \frac{X}{d^2}} \lambda(m) = \sum_{d \leqslant \sqrt{X}} \mu(d) L(X/d^2). \tag{1}$$

The equivalence of $M(X) = o(X)$ and $L(X) = o(X)$ is slightly subtle. We show that $L(X) = o(X)$ implies $M(X) = o(X)$; the argument for the other direction is almost identical. The trick is to first discard the large values of d in (1), say those with $d > X^{1/4}$ (there is a lot of flexibility here). These are bounded trivially by

$$\sum_{X^{1/4} < d \leqslant \sqrt{X}} \frac{X}{d^2} = o(X).$$

For $d \leqslant X^{1/4}$, we have $|L(X/d^2)| \leqslant \varepsilon X/d^2$ provided that X is sufficiently large in terms of X. Thus the contribution from these terms is at most

$$\sum_{d \leqslant X^{1/4}} \varepsilon \frac{X}{d^2} < C\varepsilon X.$$

Since ε was arbitrary, this is $o(X)$.

Remark. In fact, both statements are equivalent to PNT. We will see one direction of this in Question 9.

Ben Green: ben.green@maths.ox.ac.uk

8. Give an asymptotic for $\sum_{n\leqslant X}\phi(n)$. (*Hint.* Using the answer to Question 2, or otherwise, first establish that the expression to be estimated is $\sum_{d\leqslant X}\mu(d)\sum_{m\leqslant X/d} m$.)

Solution: By Question 2 we have $\phi\star 1(n)=n$. Hence, by Möbius inversion, we have

$$\phi(n)=\sum_{d|n}\mu(d)\frac{n}{d}.$$

Thus

$$\sum_{n\leqslant X}\phi(n)=\sum_{n\leqslant X}\sum_{d|n}\mu(d)\frac{n}{d}=\sum_{d\leqslant X}\frac{\mu(d)}{d}\sum_{n\leqslant X:d|n}n=\sum_{d\leqslant X}\mu(d)\sum_{m\leqslant X/d}m. \tag{2}$$

Now

$$\sum_{m\leqslant X/d}m=\frac{X^2}{2d^2}+O\Big(\frac{X}{d}\Big), \tag{3}$$

by the standard formula for triangular numbers. We substitute this into (2). The contribution from the error term in (3) to (2) is

$$\ll X\sum_{d\leqslant X}\frac{1}{d}\ll X\log X.$$

This will turn out to be tiny in comparison to contribution of the main term. The contribution of the main term $X^2/2d^2$ in (3) to (2) is

$$\frac{1}{2}X^2\sum_{d\leqslant X}\frac{\mu(d)}{d^2}.$$

Now we saw in Q1 that

$$\sum_{d}\frac{\mu(d)}{d^2}=\frac{6}{\pi^2}.$$

Since

$$\Big|\sum_{d>X}\frac{\mu(d)}{d^2}\Big|\leqslant\sum_{d>X}\frac{1}{d^2}\leqslant\int_{X-1}^{\infty}\frac{dt}{t^2}=O\Big(\frac{1}{X}\Big),$$

the main term is $\frac{3}{\pi^2}X^2+O(X)$.

Thus an asymptotic for $\sum_{n\leqslant X}\phi(n)$ is $\frac{3}{\pi^2}X^2$.

Remark. An interpretation of this is as follows: the probability that a random pair of integers $\leqslant X$ are coprime is asymptotic to $6/\pi^2$.

Ben Green: ben.green@maths.ox.ac.uk

Section C

9. The aim of this question is to show that the prime number theorem implies that $M(X) = o(X)$, where $M(X) := \sum_{n \leqslant X} \mu(n)$.

   (i) Prove that if $n \neq 1$ then
   $$-\mu(n)\log n = \sum_{ab=n} \mu(a)(\Lambda - 1)(b).$$

   (ii) Deduce that
   $$|M(X)| \leqslant \frac{1}{\log X} \sum_a \Big| \sum_{b \leqslant X/a} (\Lambda(b) - 1)\Big| + o(X).$$

   (iii) Assuming the prime number theorem, show that indeed $M(X) = o(X)$.

Solution: (i) First note that we may replace $\Lambda - 1$ with Λ away from $n = 1$, since $\sum_{ab=n} \mu(a) = 0$ unless $n = 1$.

The stated formula is then equivalent to $\mu \log = -\Lambda \star \mu$. By Möbius inversion, this is equivalent to $\Lambda = -\mu \log \star 1$. But we know that $\Lambda = \mu \star \log$; thus indeed

$$\begin{aligned}\Lambda(n) &= \sum_{d|n} \mu(d)\log(n/d) \\ &= \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d)\log d \\ &= -\sum_{d|n} \mu(d)\log d.\end{aligned}$$

(ii) We claim that

$$|M(X)| \leqslant \frac{1}{\log X}\Big| \sum_{ab \leqslant X} \mu(a)(\Lambda - 1)(b)\Big| + o(X). \tag{4}$$

By the triangle inequality it suffices to show that

$$\Big|\sum_{n \leqslant X} \mu(n)\Big(1 - \frac{\log n}{\log X}\Big)\Big| = o(X).$$

We bound this, crudely, by

$$\sum_{n \leqslant X}\Big(1 - \frac{\log n}{\log X}\Big).$$

We showed in lectures that $\sum_{n \leqslant X} \log n = X \log X + O(X)$, from which (4) follows. The bound asked for in (ii) is then immediate from the triangle inequality, using the crude bound $|\mu(a)| \leqslant 1$.

Ben Green: ben.green@maths.ox.ac.uk

(iii) This is quite similar to the last part of the preceding question. Divide the sum over a into $a \leqslant Y$ and the other values. Take $Y = X/\log X$ (again, there is some latitude in this choice). If X is large enough then, because $X/Y \to \infty$, the PNT tells us that $|\sum_{b \leqslant X/a}(\Lambda - 1)(b)| \leqslant \varepsilon \frac{X}{a}$. Thus we have

$$\frac{1}{\log X}\sum_{a \leqslant Y}\Big|\sum_{b \leqslant X/a}(\Lambda - 1)(b)\Big| \leqslant \varepsilon\frac{1}{\log X}\sum_{a \leqslant Y}\frac{X}{a} \ll \varepsilon X.$$

On the range $Y < a \leqslant X$ we have, since $\psi(X/a) \ll X/a$, the bound

$$\frac{1}{\log X}\sum_{Y < a \leqslant X}\Big|\sum_{b \leqslant X/a}(\Lambda - 1)(b)\Big| \ll \frac{X}{\log X}\sum_{Y < a \leqslant X}\frac{1}{a} = \frac{X}{\log X}(\log X - \log Y + O(1)).$$

This is $o(X)$ by the choice of Y (we need $Y = X^{1-o(1)}$ here).

10. By considering the Euler product of ζ and taking logs, show that

$$\sum_p \frac{1}{p^t} = -\log(t-1) + O(1)$$

uniformly for $t \in \mathbf{R}, t > 1$. Deduce (a weak form of Mertens' estimate) that

$$\sum_{p \leqslant X}\frac{1}{p} \ll \log\log X.$$

Solution: Start from

$$\zeta(t) = \prod_p \big(1 - \frac{1}{p^t}\big)^{-1}.$$

Taking logs gives, using $-\log(1-x) = x + O(x^2)$ uniformly for $x < 1$,

$$\log\zeta(t) = -\sum_p \log\big(1 - \frac{1}{p^t}\big) = \sum_p \frac{1}{p^t} + \sum_p O\Big(\frac{1}{p^{2t}}\Big) = \sum_p \frac{1}{p^t} + O(1). \tag{5}$$

(since $\sum_p \frac{1}{p^{2t}} \leqslant \sum_n \frac{1}{n^2}$ for all $t \geqslant 1$). Now

$$\zeta(t) = \sum_n n^{-t} = \int_1^\infty x^{-t}dx + O(1) = \frac{1}{t-1} + O(1).$$

Taking logs, we get

$$\log\zeta(t) = \log\Big(\frac{1}{t-1} + O(1)\Big) = -\log(t-1) + O(1), \tag{6}$$

since log is uniformly Lipschitz on $[1,\infty)$. Comparing (5) and (6) gives the required statement.

Ben Green: ben.green@maths.ox.ac.uk

Now take $t = 1 + \frac{1}{\log X}$, and observe that

$$\sum_p \frac{1}{p^{1+\frac{1}{\log X}}} \geqslant X^{-\frac{1}{\log X}} \sum_{p \leqslant X} \frac{1}{p} \gg \sum_{p \leqslant X} \frac{1}{p}$$

Remark. This last argument is known as 'Rankin's trick'.

Ben Green: ben.green@maths.ox.ac.uk