C3.8 Analytic Number Theory

Sheet 3 — MT24

Section A

1. Evaluate $\zeta(0)$ and $\zeta(-1)$. (*You may want to use the facts that* $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$ *and that* $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.)

   Solution: We have $\Xi(s) = \Xi(1-s)$, where $\Xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. We cannot simply substitute $s = 1$, as ζ has a pole there. Instead, set $s = 1+\varepsilon$ for some positive ε. Then

$$\zeta(0) = \lim_{\varepsilon\to 0} \frac{\pi^{-(1+\varepsilon)/2}\Gamma(\frac{1}{2}+\frac{1}{2}\varepsilon)\zeta(1+\varepsilon)}{\Gamma(-\frac{1}{2}\varepsilon)} = -\pi^{-1/2}\Gamma(\frac{1}{2})\lim_{\varepsilon\to 0}\frac{\varepsilon\zeta(1+\varepsilon)}{2\Gamma(1-\frac{1}{2}\varepsilon)} = -\frac{1}{2}\pi^{-1/2}\Gamma(\frac{1}{2}).$$

   However,

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}t^{-1/2}dt = 2\int_0^\infty e^{-x^2}dx = \pi^{1/2}.$$

   Therefore $\zeta(0) = -\frac{1}{2}$.

   Turning to $\zeta(-1)$, the functional equation tells us that

$$\zeta(-1) = \frac{\pi^{-3/2}\Gamma(1)\zeta(2)}{\Gamma(-\frac{1}{2})}.$$

   Now $\Gamma(1) = 1$, $\zeta(2) = \frac{\pi^2}{6}$, and $-\frac{1}{2}\Gamma(-\frac{1}{2}) = \Gamma(\frac{1}{2}) = \pi^{1/2}$. Putting all this together gives $\zeta(-1) = -\frac{1}{12}$.

   Remark. It is obligatory to remark that this is the famous 'formula'

$$1 + 2 + 3 + \cdots = -\frac{1}{2}.$$

   Of course this is nonsensical as written since the expression $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ is only valid for $\Re s > 1$ (and in particular not for $s = -1$).

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Section B

2. (i) Assume $\Re s > 0$. Calculate the Mellin transform $\tilde{W}(s)$, where $W(x) = 1$ for $0 < x < 1$ and $W(x) = 0$ for $x \geqslant 1$.

   (ii) Define
$$W_*(x) := \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \tilde{W}(s)x^s ds,$$
   where the integral is defined to be
$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \tilde{W}(s)x^s ds$$
   (that is, the 'Cauchy principal value' of the indefinite integral). By considering $x = 1$, show that W_* is not identically equal to W.

Solution: (i) $\tilde{W}(s) = \int_0^1 x^s \frac{dx}{x} = \left[\frac{1}{s}x^s\right]_0^1 = \frac{1}{s}$.

(ii) Parametrising the contour by $2 + it$, we have
$$W_*(1) = \frac{1}{2\pi}\int_{-T}^{T} \frac{1}{2+it}dt = \frac{1}{2\pi}\int_{-T}^{T} \frac{2-it}{4+t^2}dt.$$
The contribution from the imaginary part is, by symmetry, 0. The real part of the integral is
$$\frac{1}{\pi}\int_{-T}^{T} \frac{dt}{4+t^2} = \frac{1}{\pi}\Big[\frac{1}{2}\tan^{-1}(x/2)\Big]_{-T}^{T} \to \frac{1}{2}.$$
Remark. In fact, as it happens $W_* = W$ everywhere *except* $x = 1$. This (or rather finite truncations to large values of T) is Perron's formula. It can be used to give a "non-smooth" treatment of the PNT but one must contend with the fact that $\frac{1}{s}$ does not decay very rapidly.

3. Prove directly from the Euler product that $\zeta(s) \neq 0$ for $\Re s > 1$.

Solution: We use the Euler Product $\zeta(s) = \prod_p (1-p^{-s})^{-1}$. One must not be tempted to conclude that an infinite product of nonzero quantities is nonzero! To proceed rigorously, we use some standard techniques from the theory of infinite products. Note that by Taylor expansion we have, for $t \geqslant 0$ sufficiently small, $e^{-2t}(1+t) \leqslant 1$. It follows that if $w \in \mathbf{C}$ with $|w|$ sufficiently small then
$$e^{-2|w|}|1-w| \leqslant e^{-2|w|}(1+|w|) \leqslant 1,$$
and hence
$$|(1-w)^{-1}| \geqslant e^{-2|w|}.$$

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Applying this with $w = p^{-s}$ tells us that if $p > p_0$ is sufficiently large then

$$|(1-p^{-s})^{-1}| \geqslant e^{-2|p^{-s}|}.$$

Taking products over p, and noting that the finite product $\prod_{p \leqslant p_0}(1-p^{-s})^{-1}$ is never zero, we see that it suffices to show that $\sum_p |p^{-s}| < \infty$. But

$$\sum_p |p^{-s}| = \sum_p p^{-\Re s} < \sum_n n^{-\Re s} < \infty$$

when $\Re s > 1$.

4. Define a function $W : \mathbf{R} \to \mathbf{R}$ by

$$W(x) = \begin{cases} \exp\left(\frac{1}{x^2-1}\right) & |x| < 1 \\ 0 & |x| \geqslant 1, \end{cases}$$

Show that W is smooth.

Solution: Evidently W is smooth away from -1 and 1. By the chain rule, it is sufficient to show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(t) = e^{1/t}$ for $t < 0$ and $f(t) = 0$ for $t \geqslant 0$ is differentiable at 0. By explicit differentiation, $f^{(m)}(t) = P_m\left(\frac{1}{t}\right)e^{1/t}$ for some polynomial P_m, for $t < 0$. Thus

$$\lim_{h \to 0^-} \frac{f^{(m)}(h) - f^{(m)}(0)}{h} = \lim_{h \to 0^-} \frac{1}{h} P_m\left(\frac{1}{h}\right) e^{1/h} = 0.$$

In other words, f is infinitely differentiable at 0 and its derivative is 0 there.

5. Define functions $F_1, F_2 : \mathbf{R} \to \mathbf{R}$ by setting $F_1(x) = 1$ if $|x| \leqslant 1$, and 0 otherwise; and $F_2(x) = 1 - |x|$ if $|x| \leqslant 1$, and 0 otherwise. Show that $\int |\widehat{F}_1(\xi)| d\xi$ is infinite, but that $\int |\widehat{F}_2(\xi)| d\xi$ is finite.

Solution: We can compute the Fourier transform $\widehat{F}_1(\xi)$ quite explicitly, and compute that

$$|\widehat{F}_1(\xi)| = \frac{1}{|\xi|}|e^{2i\xi} - 1|.$$

If ξ is within $\frac{1}{10}$ of $2\pi(\mathbf{Z} + \frac{1}{2})$ then $|e^{2i\xi} - 1| > \frac{1}{2}$, so

$$\int |\widehat{F}_1(\xi)| d\xi \geqslant \frac{1}{2} \int_{\xi \in 2\pi(\mathbf{Z}+\frac{1}{2}) + B_{1/10}(0)} \frac{d\xi}{|\xi|}.$$

$$\geqslant \frac{1}{2} \sum_n \int_{2\pi(n+\frac{1}{2})-\frac{1}{10}}^{2\pi(n+\frac{1}{2})+\frac{1}{10}} \frac{d\xi}{|\xi|}.$$

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This is infinite by comparing with the harmonic series.

We can also compute the Fourier transform $\widehat{F}_2(\xi)$ explicitly. More conceptually, it is a rescaling of $F_1 * F_1$, which has Fourier transform bound by $O(|\xi|^{-2})$, which is integrable.

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6. Let $\chi : \mathbf{N} \to \{-1, 0, 1\}$ be the function defined by $\chi(n) = 0$ if $2 \mid n$, $\chi(n) = 1$ if $n \equiv 1 \pmod 4$ and $\chi(n) = -1$ if $n \equiv 3 \pmod 4$.

   (i) Show that χ is completely multiplicative.

   (ii) Define $L(s, \chi) := \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1}$. Evaluate $\lim_{s\to 1^+} L(s, \chi)$.

   (iii) Deduce that $\lim_{s\to 1^+} \sum_p \chi(p)p^{-s}$ converges.

   (iv) Conclude that there are infinitely many primes congruent to 1 mod 4, and also infinitely many primes congruent to 3 mod 4.

Solution: (i) is easily checked by looking at cases modulo 4. (What is really going on here is that χ is induced from a homomorphism on $(\mathbf{Z}/4\mathbf{Z})^\times$.)

(ii) The product is

$$\prod_p \left(1 + \chi(p)p^{-s} + \chi(p^2)p^{-2s} + \dots\right) = \sum_n \chi(n)n^{-s},$$

since χ is completely multiplicative. Taking limits as $s \to 1^+$ gives

$$\lim_{s\to 1^+} L(s, \chi) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

(iii) We have $1 - \chi(p)p^{-s} = e(-\chi(p)p^{-s}) + O(p^{-2s})$. Thus, since $\lim_{s\to 1^+} L(s, \chi)$ is bounded and not zero, it follows that

$$\lim_{s\to 1^+} \sum_p \chi(p)p^{-s}$$

is finite.

(iv) We have

$$\lim_{s\to 1^+} \sum_{p\equiv 1 (\bmod 4)} p^{-s} = \frac{1}{2} \lim_{s\to 1^+} \sum_p \frac{1 + \chi(p)}{p^s},$$

which is infinite since $\lim_{s\to 1^+} \sum \frac{1}{p^s}$ is (this follow from Sheet 2, Q10). Similarly

$$\lim_{s\to 1^+} \sum_{p\equiv 3 (\bmod 4)} \frac{1}{p^s} = \frac{1}{2} \lim_{s\to 1^+} \sum_p \frac{1 - \chi(p)}{p^s}$$

is infinite.

7. Show that $\zeta(s)$ does not vanish for real s in the interval $[0, 1]$.

Solution: From lectures,

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx.$$

Hence if $s > 0$ and $\zeta(s) = 0$ then

$$\frac{1}{s-1} = \int_1^\infty \frac{\{x\}}{x^{s+1}}dx.$$

However, the LHS is negative, whilst the RHS is positive. This is a contradiction.

We showed that $\zeta(0) \neq 0$ in the first question.

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Section C

8. Construct a smooth function $F : \mathbf{R} \to \mathbf{R}$ such that $F(x) = 1$ when $|x| \leqslant 1$, $0 \leqslant F(x) \leqslant 1$ when $1 \leqslant |x| \leqslant 2$, and $F(x) = 0$ when $|x| \geqslant 2$.

   Solution: By Question 4 and a rescaling, there is a nonnegative smooth function g supported on $|x| \leqslant \frac{1}{2}$ and with $\int g = 1$. Define

$$F(x) := \int_{-3/2}^{3/2} g(x+t)dt$$

   (thus F is basically the convolution of $1_{[-3/2,3/2]}$ with g.) It is easy to see that F has all the stated support properties. For example, if $|x| \leqslant 1$ then $F(x) = \int_{-\infty}^{\infty} g(x+t)dt = \int g = 1$, since the portions of the integral with $|t| > \frac{3}{2}$ contribute nothing.

   Moreover, F is smooth. This follows from standard results about differentiating under integrals; it is also not hard to reproduce an argument directly, which we shall now do. It suffices (by induction) to show that

$$\tilde{F}(x) = \int_{-3/2}^{3/2} g'(x+t)dt$$

   is a derivative of F. We have

$$F(x+h) - F(x) - h\tilde{F}(x) = \int_{-3/2}^{3/2} \big(g(x+h+t) - g(x+t) - hg'(x+t)\big)dt.$$

   However by MVT

$$g(x+h+t) - g(x+t) - hg'(x+t) = h\big(g'(x+t+\theta_{x,t,h}) - g'(x+t)\big),$$

   where $0 \leqslant \theta_{x,t,h} \leqslant h$, and this is, by MVT again, bounded by $O(|h|^2\|g''\|_\infty)$. It follows that

$$F(x+h) - F(x) - h\tilde{F}(x) \ll |h|^2\|g''\|_\infty \ll |h|^2.$$

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9. Suppose that Ω is a countable multiset of nonzero complex numbers, and that

$$\sum_{\rho\in\Omega} |\rho|^{-2} < \infty. \tag{1}$$

Explain how the infinite sum

$$F(s) := \sum_{\rho\in\Omega} \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big)$$

may be defined rigorously, and why it is meromorphic except for poles at the ρ.

Solution: First observe that the condition (1) implies that Ω has only finitely many points in any open ball in $\mathbf{C}$. Suppose first that $s_0 \notin \Omega$. Then, by the property just mentioned, there is some ball $B_\varepsilon(s_0)$, $\varepsilon > 0$, disjoint from Ω. For $s \in B_\varepsilon(s_0)$, and for $N > 2|s|$, define

$$F_N(s) := \sum_{\rho\in\Omega, |\rho|\leqslant N} \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big).$$

Suppose that $N' > N$. Repeating (essentially) a computation from Chapter 7 of lectures, we have

$$\begin{aligned} |F_N(s) - F_{N'}(s)| \leqslant \sum_{\rho\in\Omega, N<|\rho|\leqslant N'} \Big|\frac{1}{s-\rho} + \frac{1}{\rho}\Big| &= |s| \sum_{\rho\in\Omega, N<|\rho|\leqslant N'} \frac{1}{|s-\rho||\rho|} \\ &\leqslant 2|s| \sum_{\rho\in\Omega, N<|\rho|\leqslant N'} |\rho|^{-2}. \end{aligned}$$

(Here, as in lectures, we used the fact that $|\rho| \geqslant 2|s|$, so $|s/\rho| \leqslant \frac{1}{2}$, and hence $|\frac{s}{\rho} - 1| \geqslant \frac{1}{2}$ and hence $|s-\rho| \geqslant \frac{1}{2}|\rho|$.) It follows that $F_N(s)$ is uniformly Cauchy on $B_\varepsilon(s_0)$, and hence converges uniformly to a holomorphic function on this domain. This limit is the desired rigorous definition of $F(s)$.

Around any s which is equal to some ρ, a very similar argument applies, first setting aside the term $\frac{k}{s-\rho}$, where k is the multiplicity of ρ. We leave the details as an exercise.

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