C3.8 Analytic Number Theory

Sheet 4 — MT24

Section A

1. As usual, write $\psi(X) := \sum_{n \leqslant X} \Lambda(n)$, and write $\pi(X)$ for the number of primes less than or equal to X. Show that

$$\pi(X) = \int_0^1 \psi(\min(e^{1/t}, X))dt + O(X^{1/2}).$$

Assuming the prime number theorem with classical error term, deduce that

$$\pi(X) = \int_2^X \frac{dt}{\log t} + O(Xe^{-c\sqrt{\log X}})$$

for some absolute constant $c > 0$.

Solution: We have

$$\pi(X) = \sum_{n \leqslant X} \frac{\Lambda(n)}{\log n} + O(X^{1/2}),$$

the error term coming from the prime powers which are not primes. (More carefully: the contribution from these is bounded by the number of such prime powers less than or equal to X, which is at most $\sum_{p \leqslant X^{1/2}} \lfloor \frac{\log X}{\log p} \rfloor$. The contribution from $p \leqslant X^{1/4}$ is really tiny, and the contribution from $X^{1/4} \leqslant p \leqslant X^{1/2}$ is $O(1)$ times the number of primes in this range, which is definitely $\ll X^{1/2}$.)

Turning to the question, we claim that

$$\sum_{n \leqslant X} \frac{\Lambda(n)}{\log n} = \int_0^1 \psi(\min(e^{1/t}, X))dt. \tag{1}$$

This is a simple matter of changing the order of integration; the RHS is

$$\int_0^1 \Big(\sum_{\substack{n \leqslant e^{1/t} \\ n \leqslant X}} \Lambda(n) \Big) dt = \sum_{n \leqslant X} \Lambda(n) \Big(\int_0^1 \sum_{n \leqslant e^{1/t}} dt \Big) = \sum_{n \leqslant X} \Lambda(n) \Big(\frac{1}{\log n} \Big).$$

Assuming the PNT with classical error term, we have

$$\psi(\min(e^{1/t}, X)) = X + O(Xe^{-c\sqrt{\log X}}) \tag{2}$$

for $t \leqslant 1/\log X$ (that is, when $\min(e^{1/t}, X) = X$), and

$$\psi(\min(e^{1/t}, X)) = e^{1/t} + O(e^{1/t - c/t^{1/2}}) \tag{3}$$

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s if $t \geqslant 1/\log X$. We substitute these estimates in to the RHS of (1), looking first at the main terms. The contribution from $t \leqslant 1/\log X$ is evidently $X/\log X$. The contribution from $t \geqslant 1/\log X$ is (making the substituting $t = 1/\log u$ in the integral)

$$\int_{1/\log X}^{1} e^{1/t} dt = \int_e^X \frac{du}{(\log u)^2}.$$

However by integration by parts

$$\int \frac{du}{\log u} = \frac{u}{\log u} + \int \frac{du}{(\log u)^2},$$

so

$$\int_e^X \frac{du}{(\log u)^2} = \int_e^X \frac{du}{\log u} - \frac{X}{\log X} + e.$$

It follows that the total contribution to (1) from the main terms in (2), (3) is

$$\int_2^X \frac{du}{\log u} + O(1).$$

Turning now to the contribution of the error terms in (2), (3), the contribution from $t \leqslant 1/\log X$ is evidently $\ll X e^{-c\sqrt{\log X}}$ (which is what we are aiming for). Finally, the contribution from $t \geqslant 1/\log X$ is bounded by

$$\int_{1/\log X}^{1} e^{1/t - c/t^{1/2}} dt.$$

Assuming $c < 1$ (which, without loss of generality, we may), $1/t - c/t^{1/2}$ is a decreasing function of t in $(0, 1)$. Therefore the integral is bounded by its value at $t = 1/\log X$, which is $X e^{-c\sqrt{\log X}}$.

2. Show that for any integer $n \geqslant 2$ we have

$$\int_2^X \frac{dt}{\log t} = \frac{X}{\log X} + \frac{X}{(\log X)^2} + \cdots + (n-1)! \frac{X}{(\log X)^n} + O_n\Big(\frac{X}{(\log X)^{n+1}}\Big).$$

Solution: This follows by repeated integration by parts, together with the estimate

$$\int_2^X \frac{dt}{(\log t)^{n+1}} \ll_n \frac{X}{(\log X)^{n+1}}$$

for the last term, which follows by splitting the range of integration into $X^{1/2} < t \leqslant X$ and $t \leqslant X^{1/2}$ (say).

Section B

3. Prove that $\pi(X) \leqslant 2\pi(X/2)$ for X sufficiently large (you may use the results of Questions 1 and 2 if you want).

Solution: By Questions 1 and 2 we have we have

$$\pi(X) = \frac{X}{\log X} + \frac{X}{\log^2 X} + O\Big(\frac{X}{\log^3 X}\Big)$$

and

$$2\pi(X/2) = \frac{X}{\log X - \log 2} + \frac{X}{\log^2 X} + O\Big(\frac{X}{\log^3 X}\Big).$$

But

$$\frac{X}{\log X - \log 2} = \frac{X}{\log X}\Big(1 + \frac{\log 2}{\log X}\Big) + O\Big(\frac{1}{\log^3 X}\Big).$$

Since $\log 2 < 1$, the result follows.

4. Suppose that there were no nontrivial zeros of ζ. Let W be any smooth function supported on $[1, 2]$, with $\|W\|_\infty \leqslant 1$. Show that for $X \geqslant 2$ we have

$$\sum_n \Lambda(n) W\Big(\frac{n}{X}\Big) = X\Big(\int W\Big) + O(X^{-2}),$$

the error term being uniform in the choice of W. Use this to derive a contradiction, thereby concluding that ζ does have a nontrivial zero.

Solution: From the explicit formula we have

$$\sum_n \Lambda(n) W\Big(\frac{n}{X}\Big) = X\Big(\int W\Big) - \sum_{j=1}^{\infty} X^{-2j} \tilde{W}(-2j).$$

Now

$$\tilde{W}(-2j) = \int_1^2 W(x) x^{-2j} \frac{dx}{x} \leqslant 1,$$

and therefore

$$\Big|\sum_n \Lambda(n) W(\frac{n}{X}) - X\Big| \leqslant 2 \sum_j X^{-2j} \ll X^{-2},$$

uniformly in the choice of W. Note in particular that the error term does *not* depend on any smoothness norms of W.

Fix X, and let W_ε be a smooth function equal to 1 on $[1 + \varepsilon, \sqrt{2} - \varepsilon]$ and supported on $[1, 2]$ (all that matters here is that $\sqrt{2}$ is irrational). Then

$$\sum_n \Lambda(n) W\Big(\frac{n}{X}\Big) = X\Big(\int W_\varepsilon\Big) + O(X^{-2}).$$

As $\varepsilon \to 0$, the RHS tends to $X + O(X^{-2})$, whilst the LHS tends to

$$\sum_{X < n < X\sqrt{2}} \Lambda(n).$$

Thus

$$\sum_{X < n < X\sqrt{2}} \Lambda(n) = X(\sqrt{2} - 1) + O(X^{-2}).$$

The right-hand side $R(X)$ here is almost continuous as a function of X; in particular $R(X + \frac{1}{X^2}) - R(X) = O(X^{-2})$. By contrast the left-hand side $L(X)$ is not; in fact, it has jumps of size $\sim \log X$ when X passes through a prime power. This is the desired contradiction.

5. Assume that all zeros of ζ are simple. Let W be a smooth, compactly supported function supported on $(0, \infty)$, and suppose that X is sufficiently large. Give an explicit formula for

$$\sum_n \mu(n) W\Big(\frac{n}{X}\Big).$$

(I do not expect you to write down any details: assume that analogues of all the estimates we established in deriving the explicit formula for Λ hold here also.)

Solution: By following the sketch proof of the explicit formula itself in lectures, and recalling that the Dirichlet series of Mobius is $1/\zeta(s)$, we see that

$$\sum_n \mu(n) W\Big(\frac{n}{X}\Big) = \sum_\rho \mathrm{Res}_{s=\rho} \frac{1}{\zeta(s)} X^s \tilde{W}(s), \tag{4}$$

where the sum is over all zeros of ζ, which of course correspond precisely to the poles of $1/\zeta$. By assumption these are all simple. By Taylor expansion,

$$\lim_{s \to \rho} \frac{\zeta(s)}{s - \rho} = \zeta'(\rho),$$

and so

$$\mathrm{Res}_{s=\rho} \frac{1}{\zeta(s)} X^s \tilde{W}(s) = \frac{1}{\zeta'(\rho)} X^\rho \tilde{W}(\rho).$$

Substituting in to (4) gives the required explicit formula.

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6. Let Ω be a countable multiset of nonzero complex numbers, and suppose that $\sum_{\rho\in\Omega}|\rho|^{-2} < \infty$. Consider the Weierstrass product $f(z) = \prod_\rho \big(1 - \frac{z}{\rho}\big)e^{z/\rho}$. Show rigorously that
$$\frac{f'(z)}{f(z)} = \sum_\rho \Big(\frac{1}{z-\rho} + \frac{1}{\rho}\Big)$$
if $z \notin \Omega$. (Hint: you may use Sheet 3, Q9 as well as any part of the proof of Proposition C.2 in the notes that you want. It may be helpful to recall Cauchy's formula for derivatives).

Solution: Consider the partial product
$$f_N(z) := \prod_{|\rho|\leqslant N}(1 - \frac{z}{\rho})e^{z/\rho}. \tag{5}$$
We showed in Appendix C of the notes that $f_N(z) \to f(z)$ uniformly on compacta. As it happens, this already implies that $f_N'(z) \to f'(z)$, uniformly on compacta. Indeed by Cauchy's formula for derivatives, if $|z| = K$ then
$$f_N'(z) - f(z) = \frac{1}{(2\pi i)^2}\int_{|w|=2K}\frac{f_N(w) - f(w)}{(w-z)^2}dw \ll K^{-1}\sup_{|w|=2K}|f_N(w) - f(w)|.$$
Evidently (by differentiation of (5)) if $z \notin \Omega$ then
$$\frac{f_N'(z)}{f_N(z)} = \sum_{|\rho|\leqslant N}\Big(\frac{1}{z-\rho} + \frac{1}{\rho}\Big).$$
The result now follows by letting $N \to \infty$: by the remarks above, the LHS tends to $f'(z)/f(z)$, while by Sheet 3 Q9, the right hand side tends to $\sum_\rho \Big(\frac{1}{z-\rho} + \frac{1}{\rho}\Big)$ (indeed, it is the definition of that quantity).

7. In this question, we assume that there is a compactly supported non-zero smooth function W satisfying $|\tilde{W}(\rho)| \ll e^{-c|t|^{1/2}}$ for every nontrivial zero $\rho = \sigma + it$ (such a function can be constructed by an elaboration of the methods of Appendix A of the notes). Using the classical zero-free region (that is, every nontrivial zero satisfies $\sigma < 1 - \frac{c}{\log(|t|+2)}$), obtain as good a bound as you can for the error term $\sum_n \Lambda(n)W(\frac{n}{X}) - X\big(\int W\big)$, as $X \to \infty$.

Solution: We use the explicit formula, which gives an expression for the error term in terms of a sum over zeros. The contribution from the trivial zeros is tiny as usual. The contribution from the nontrivial zeros is bounded by
$$\sum_\rho X^{\sigma+it}e^{-c|t|^{1/2}},$$

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which is bounded by
$$\sum_{\rho} X^{1-\frac{c}{\log(2+|t|)}} e^{-c|t|^{1/2}}. \tag{6}$$

To estimate this sum, we divide into dyadic ranges $[T, 2T]$ according to the size of the imaginary part t of ρ. We will use Proposition 6.8 from lectures, which tells us that at most $O(T \log T)$ zeros ρ have imaginary part in this range.

Suppose first that $T > C(\log X)^2$. Then, bounding $X^{-\frac{c}{\log(2+|t|)}}$ trivially by 1, the sum over this range is bounded by

$$\ll X(T\log T)e^{-c|T|^{1/2}} < X^{-10}T^{-2}$$

(if C is big enough). Summing (with T ranging over powers of two) we see that the total contribution to (6) from $|t| > C(\log X)^2$ is $\ll X^{-10}$, which is negligible.

For the remaining range $|t| \leqslant C(\log X)^2$, we instead bound $e^{-c|t|^{1/2}}$ trivially by 1, and use the fact that
$$\frac{\log X}{\log(2+|t|)} \gg \frac{\log X}{\log\log X}$$
on this range.

A crude estimate for the number of zeros involved is $O((\log X)^3)$, so we obtain a bound for (6) of
$$\ll X(\log X)^3 e^{-c_1\frac{\log X}{\log\log X}} \ll X e^{-c_2\frac{\log X}{\log\log X}}$$

(for some absolute constants c_1, c_2). *Remark.* The point here is that this error term is quite a bit better than the one in the prime number theorem, due to the use of the smooth weight function W.

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Section C

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8. Suppose that ρ lies in the critical strip $0 \leqslant \Re\rho \leqslant 1$. Write $s = 2 + iT$. Show that

$$\Re\Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \geqslant 0$$

and that

$$\Re\Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \geqslant \frac{1}{5}$$

if $|\Im\rho - T| \leqslant 1$.

By using this and the partial fraction expansion of ζ'/ζ, prove that the number of nontrivial zeros with $|\Im\rho - T| \leqslant 1$ is $O(\log T)$ (for T large).

Solution: We start with the first statement. Write $\rho = \beta + i\gamma$. Then $\Re(1/\rho) \geqslant 0$ so we can ignore this. We also have

$$\Re\frac{1}{s-\rho} = \frac{2-\beta}{(2-\beta)^2 + (T-\gamma)^2} \geqslant 0.$$

The first statement follows.

For the second statement we use the same bound but now note that $2 - \beta \geqslant 1$, that $2 - \beta \leqslant 2$ and, due to the assumption, that $|T - \gamma| \leqslant 1$. The second statement follows.

For the final statement, suppose that the number of $\rho \in Z_{\text{nontriv}}$ with $|\Im\rho - T| \leqslant 1$ is M. Then, from the above, with $s = 2 + iT$

$$\Big|\sum_{\rho\in Z_{\text{nontriv}}} \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big)\Big| \geqslant \frac{M}{5}.$$

On the other hand,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_n \Lambda(n) n^{-s} = O(1)$$

uniformly in T, whilst

$$\Big|\frac{1}{s-1}\Big| = O(1),$$

also uniformly in T. Therefore from the partial fraction expansion of $\zeta'(s)/\zeta(s)$ we have

$$\Big|\sum_{\rho\in Z_{\text{triv}}} \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big)\Big| \geqslant \frac{M}{5} - O(1). \tag{7}$$

But with $s = 2 + iT$ and $\rho = -2k$ we have

$$\frac{1}{s-\rho} + \frac{1}{\rho} = \Big|\frac{2+iT}{2k(2+iT+2k)}\Big| \ll \frac{T}{k(k+T)}.$$

The sum of this over $k \leqslant T$ is $O(\log T)$, whilst the sum over $k > T$ is $\ll \sum_{k>T} \frac{T}{k^2} = O(1)$. Thus

$$\Big|\sum_{\rho\in Z_{\text{triv}}} \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big)\Big| = O(\log T).$$

Comparing this with (7) we obtain $M \ll \log T$, which is the required result.

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