Problem Sheet 2

Section A

Question 1. **Mollification.**

(1) Give an example of a function (that will play later the role of kernel for mollification) with the following properties:
- $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\text{supp}(\phi) = B_1(0)$;
- $\phi(x) \geq 0$ for all $x \in \mathbb{R}^n$;
- $\int_{B_1(0)} \phi(x)dx = 1$.

(2) Given ϕ as in point 1, for every function $u \in L^1_{loc}(\mathbb{R}^n)$ define
$$u \star \phi(x) := \int_{\mathbb{R}^n} u(x-y)\phi(y)dy.$$
Show that $u \star \phi \in C^\infty(\mathbb{R}^n)$.
Hint: observe that $\int_{\mathbb{R}^n} u(x-y)\phi(y)dy = \int_{\mathbb{R}^n} \phi(x-y)u(y)dy$.

(3) Given ϕ as in point 1, for every $\epsilon \in (0,1)$, let
$$\phi_\epsilon(x) := \epsilon^{-n}\phi(x/\epsilon).$$
Show that $\text{supp}(\phi_\epsilon) = B_\epsilon(0)$ and that $\int_{B_\epsilon(0)} \phi_\epsilon(x)dx = 1$.

(4) If $u \in C(\mathbb{R}^n)$, show that $u \star \phi_\epsilon$ converges to u uniformly on compact subsets of $\mathbb{R}^n$.

Solution For more on mollification see the Lecture Notes of C4.3 "Functional Analytic methods for PDEs".

(1) Define $\phi(x) := 0$ for $|x| \geq 1$ and $\phi(x) := C\exp(\frac{1}{|x|^2-1})$ for $|x| < 1$, with $C > 0$ chosen such that $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. It is easily seen that such ϕ has all the desired properties.

(2) First of all we notice that, if $u \in L^1_{loc}(\mathbb{R}^n)$ then
$$u \star \phi(x) := \int_{\mathbb{R}^n} u(x-y)\phi(y)dy$$
is well defined for all $x \in \mathbb{R}^n$. By the change of variable $z = x - y$ we directly see that
$$u \star \phi(x) := \int_{\mathbb{R}^n} u(x-y)\phi(y)dy = \int_{\mathbb{R}^n} u(z)\phi(x-z)dz = \int_{\mathbb{R}^n} \phi(x-y)u(y)dy,$$
proving the hint. Now, since ϕ is C^1 with compact support, we can use the Differentiation Theorem (it is a corollary of Dominated Convergence Theorem) to infer that
$$\partial_{x_i}(u \star \phi)(x) = \partial_{x_i}\left(\int_{\mathbb{R}^n} \phi(x-y)u(y)dy\right) = \int_{\mathbb{R}^n} (\partial_{x_i}\phi)(x-y)u(y)dy.$$
This shows that $u \star \phi$ is C^1. By iterating the procedure, we obtain that $u \star \phi$ is C^∞.

(3) Follows directly from (1) by changing variables.

(4) If $u \in C(\mathbb{R}^n)$ then it is uniformly continuous on compact subsets. Fix a compact subset $K \Subset \mathbb{R}^n$. Using that $\phi_\epsilon \geq 0$, $\int \phi_\epsilon = 1$ and that $\text{supp}(\phi_\epsilon) = B_\epsilon(0)$, for every $x \in K$ we have that
$$|u(x) - u \star \phi_\epsilon(x)| = \left|\int_{\mathbb{R}^n} (u(x) - u(x-y))\,\phi_\epsilon(y)dy\right| \leq \int_{\mathbb{R}^n} |u(x) - u(x-y)|\,\phi_\epsilon(y)dy$$
$$\leq \sup_{y \in \mathbb{R}^n, |y| \leq \epsilon} |u(x) - u(x-y)|\,.$$
Denote with $K_1 := \{x \in \mathbb{R}^n : \text{there exists } y \in K \text{ such that } |x-y| \leq 1\}$, (i.e. the set of points at distance at most 1 from K) and notice that K_1 is compact as well. From the previous estimate, we obtain
$$\sup_{x \in K} |u(x) - u \star \phi_\epsilon(x)| \leq \sup_{x_1, x_2 \in K_1, |x_1 - x_2| \leq \epsilon} |u(x_1) - u(x_2)| \to 0 \quad \text{as } \epsilon \to 0$$
by uniform continuity of u on the compact set K_1.

□

QUESTION 2. **An application of Brouwer's fixed point Theorem: zero's of continuous vector fields.**

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. Assume that there exists $R > 0$ such that

$$g(x) \cdot x \geq 0, \qquad \text{for all } x \text{ with } |x| = R.$$

Show that there exists $x^* \in \overline{B_R(0)}$ such that $g(x^*) = 0$; in other words, show that the vector field g has a zero in $\overline{B_R(0)}$.

Hint: Argue by contradiction, consider the map $f(x) := -\frac{R}{|g(x)|} g(x)$ *and apply Brouwer's fixed point Theorem.*

Solution. Assume that there exists no such x^*. Then we can define

$$f(x) = -R\frac{g(x)}{|g(x)|}.$$

f is continuous and $f\colon \overline{B_R(0)} \to \overline{B_R(0)}$. Brouwer's FPT implies that there exists $x_1 \in \overline{B_R(0)}$ such that $f(x_1) = x_1$. Then $|x_1| = |f(x_1)| = R$, and thus the assumption on g implies $g(x_1) \cdot x_1 \geq 0$.

On the other hand

$$g(x_1) \cdot x_1 = -f(x_1) \cdot x_1 \frac{|g(x_1)|}{R} = -\frac{|x_1|^2 \, |g(x_1)|}{R} < 0,$$

which is a contradiction. □

Section C

QUESTION 6. **Leray's eigenvalue problem**. Let $K : [a, b] \times [a, b] \to (0, \infty)$ be a continuous and positive function and consider the integral operator $T : C^0([a, b]) \to C^0([a, b])$ defined by

$$(Tu)(x) = \int_a^b K(x, t)u(t)\, dt.$$

Prove that T has at least one non-negative eigenvalue λ whose eigenvector is a continuous non-negative function u, i.e. there exist $\lambda \geq 0$ and a non-negative u so that

$$\int_a^b K(x, t)u(t)\, dt = \lambda u(x).$$

Hint: consider, on an appropriate closed convex set M, the function

$$F(u) = \frac{1}{\int_a^b Tu(t)dt} \cdot Tu.$$

and apply one of the versions of Schauder's Fixed Point Theorem with the help of Arzéla-Ascoli Theorem. To find a suitable set M think about what property all functions $F(u)$ have in common.

Solution. Since $K : [a, b] \times [a, b] \to (0, \infty)$ is continuous, there exist $c_1, c_2 \in (0, \infty)$ such that

$$c_1 \leq K(x, t) \leq c_2\,, \quad \text{for all } (x, t) \in [a, b]^2.$$

We know from First year Analysis that if $u \in C^0([a, b])$, then the function $x \mapsto \int_a^b K(x, t)u(t)\, dt := Tu(x)$ is continuous on $[a, b]$ as well. Moreover, if $u \geq 0$ then we have

$$c_1 \int_a^b u(t)\, dt \leq \int_a^b K(x, t)u(t)\, dt \leq c_2 \int_a^b u(t)\, dt, \quad \text{for all } u \geq 0.$$

Consider now

$$F(u) := \frac{1}{\int_a^b Tu(t)dt} \cdot Tu.$$

Observe that $\int_a^b (Fw)(t)\, dt = 1$ for every $w \geq 0$. Then, any fixed point of F will satisfy $u(x) = (Fu)(x)$ so in particular

$$\int_a^b u(t)\, dt = \int_a^b (Fu)(t)\, dt = 1.$$

Observe that

$$M := \left\{ u \in C^0([a, b]) \,:\, u \geq 0, \int_a^b u(t)\, dt = 1, \right\}$$

is convex, closed and non-empty. In order to apply Schauder Theorem version III, we need to prove that $F : M \to M$ is continuous and that $F(M)$ is compact.

Claim 1: $F : M \to M$ is continuous.

We know that $Tu(x) \geq c_1(b - a) > 0$. It easily follows that $Fu(x) \geq 0$ for all $x \in [a, b]$ as well. Moreover, we already observed that $\int_a^b (Fu)(t)\, dt = 1$, and thus F maps M to M.

Proof that $F : M \to M$ is continuous. since the map $u \mapsto K(u)$ is continuous on $C^0([a, b])$, so also the map $u \mapsto \int_a^b K(u)(t)\, dt$ is continuous. Moreover, this is bounded from below:

$$0 < c_1|b - a|^2 \leq \int_a^b Tu(t)\, dt \leq c_2|b - a|^2. \tag{1}$$

So claim 1 follows.

Claim 2: $F(M)$ is compact.

We first show that $F(M)$ is bounded. From (1) we obtain that

$$0 \leq \frac{c_1}{c_2(b-a)} \leq F(u)(t) \leq \frac{c_2}{c_1(b-a)}$$

for all $u \in M$ and all $t \in [a,b]$. Thus $F(M)$ is bounded.

In order to show that $F(M)$ is pre-compact it is then enough to show that it is equi-continuous (so the pre-compactness will follow from Arzelá-Ascoli's Theorem).

Let $\delta > 0$, $t_{1,2}$ be such that $|t_1 - t_2| < \delta$ and let $u \in M$. Denote $\mu(u) := \int_a^b K(u)(t)\, dt$. Then

$$|F(u)(t_1) - F(u)(t_2)| \leq \frac{1}{\mu(u)} \int |K(t_1,x) - K(t_2,x)|\, u(x)\, dx \leq \frac{1}{\mu(u)} \sup_{x, |t_1 - t_2| < \delta} |K(t_1,x) - K(t_2,x)|.$$

The conclusion follows by the uniform continuity of K on the compact set $[a,b]^2$ and by lower bound $\mu(u) \geq (b-a)^2 c_1 > 0$. □