Geometric Group Theory

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Some inspirational quotations

Herman Weyl: "In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain."

Guillermo Moreno: "Groups, as men, will be known by their actions."

Resources

- Lecture notes.
- Problem Sheet 0 and Solutions.
- Four other Problem sheets.
- Hand-out Notes with material from previous years.
- Books: J.P. Serre - Trees
  D. – Kapovich: Geometric Group Theory (Chapters 7,8,11).

Plan of the course

We will be studying countable infinite groups. Unless otherwise stated, all groups will be finitely generated.

We have two types of infinite groups:

- "Small": Infinite abelian $\subset$ Nilpotent $\subset$ Polycyclic $\subset$ Solvable (studied in MT, C2.4 Infinite Groups).
- "Large": Free groups $\subset$ Hyperbolic groups; Free groups $\subset$ Amalgamated products.

Methods used

(1) Geometric Approach: Make the group G act on an interesting metric space X. Deduce algebraic properties of G from

- the geometry of X
- the properties of the action of G on X

For example, X might be a Hilbert space (in particular $\mathbb{R}^n$) or a differentiable/Riemannian manifold. In this course, we consider the case when X is a simplicial tree and, later, when $X = \mathbb{H}^2_{\mathbb{R}}$ or $X = \mathbb{H}^n_{\mathbb{R}}$.

(2) Algorithmic Approach: Design algorithms and construct Turing machines to solve algebraic questions. This works for groups described by finite data (i.e. finitely presented groups). Dehn (1912) formulated 3 fundamental problems:

I Word Problem

II Conjugacy Problem

III Isomorphism Problem

Methods used

Dehn solved the three problems for G acting on $\mathbb{H}^2$ by isometries, $G \leq \mathrm{Isom}(\mathbb{H}^2)$ discrete and $\mathbb{H}^2/G$ compact.

However, the problems are unsolvable in general (Novikov–Boone). The example of a group with unsolvable word problem that Novikov–Boone constructed acts on a tree.

(3) Approximation by finite groups: The idea is to take finite quotients $G/N_k, k \in \mathbb{N}$ that become larger and larger. Ideally $\bigcap_{k\in\mathbb{N}} N_k = \{1\}$.

- Residually finite groups.

Methods used

(4) Understanding of subgroups:

- What kind of subgroups?
- Can we decompose every group G into simpler "building blocks"?
- "Small" groups: basic building blocks are abelian; the general groups obtained by iterating semidirect products, more generally short exact sequences;
- "Large" groups: we iterate amalgamated products and HNN extensions;building blocks are one-ended groups etc.;

Methods used

The main trick of GGT is to move between:

Discrete world :

- Graphs

Toolkit: Combinatorics, discrete maths

Continuous world:

- Differentiable manifolds $\supseteq$ Riemannian manifolds

Toolkit: Calculus

Dehn used the geometry of a continuous space (the real hyperbolic plane) to solve a problem about a discrete group. This can be done backwards: use a group/discrete structure to solve a problem about a continuous space.

Generating sets

Given $S \subset G$ and $H \leq G$, TFAE

- H is the smallest subgroup of G containing S;
- $H = \bigcap_{S \subset K \leq G} K$;
- $H = \{s_1^{\pm 1} s_2^{\pm 1} \dots s_n^{\pm 1} \mid n \in \mathbb{N}, s_i \in S\} \cup \{\mathrm{id}\}$.

H is called the subgroup generated by S. We write $H = \langle S \rangle$.

- If $H = G$ then S is called a generating set.
- If S finite then G is called finitely generated.
- If $S = \{x\}$ then $\langle x \rangle$ is the cyclic subgroup generated by x.
- Rank of G= minimal number of generators.

Free groups

What is "the largest infinite group" generated by n elements?

Finite sets: A larger than $B \Leftrightarrow \mathrm{card}(A) \geq \mathrm{card}(B) \Leftrightarrow$ there exists $f : A \to B$ onto.

Infinite groups: We look for a group $G = \langle X \rangle$, $\mathrm{card}(X) = n$, such that for every group $H = \langle Y \rangle$, $\mathrm{card}(Y) = n$, a bijection $X \to Y$ extends to an onto group homomorphism.

Clearly cannot be done for any group G, e.g. if G is abelian then H would have to be abelian.

So G must be a group with no prescribed relation ("free").

Construction of a free group

$X \neq \emptyset$. Its elements = letters/symbols.

Take inverse letters/symbols $X^{-1} = \left\{a^{-1} \mid a \in X\right\}$.

We call $\mathcal{A} = X \sqcup X^{-1}$ an alphabet.

A word w in $\mathcal{A}$ is a finite (possibly empty) string of letters in $\mathcal{A}$

$$a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_k}^{\epsilon_k},$$

where $a_i \in X, \epsilon_i = \pm 1$.

The length of w is k.

We use the notation 1 for the empty word (the word with no letters). We say it has length 0.

Construction of a free group 2

A word w is reduced if it contains no pair of consecutive letters of the form aa^{-1} or $a^{-1}a$.

A reduction of a word w is the deletion of a pair of consecutive letters of the form aa^{-1} or $a^{-1}a$.

An insertion is the opposite operation: insert a pair of consecutive letters of the form aa^{-1} or $a^{-1}a$.

Denote by X^* the set of words in the alphabet $\mathcal{A} = X \sqcup X^{-1}$, empty word included.

Denote by $F(X)$ the set of reduced words in $\mathcal{A}$, empty word included.

We define an equivalence relation on X^* by $w \sim w'$ if w' can be obtained from w by a finite sequence of reductions and insertions.

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Proposition

$\forall w \in X^*$, there exists a unique $u \in F(X)$ such that $w \sim u$.

Proof

Existence: By induction on the length.

$$\begin{aligned} w = a_1 a_2 \ldots a_{n+1} &\sim a_1 b_1 \ldots b_k \quad \text{if } a_1 b_1 \neq x x^{-1}, x^{-1} x \\ &\sim b_2 \ldots b_k \quad \text{otherwise} \end{aligned}$$

Construction of a free group 4

Uniqueness: We prove that if $u, v \in F(X)$, $u \neq v$, then we cannot have $u \sim v$.

Argue by contradiction and assume we can. So there exists a sequence of reductions and insertions

$$w_0 = u \sim w_1 \sim w_2 \sim \ldots \sim w_{n-1} \sim w_n = v$$

Take a sequence with $\sum |w_i|$ minimal. As u and v are reduced, $w_0 \to w_1$ is an insertion and $w_{n-1} \to w_n$ is a reduction. Hence $|w_0| < |w_1|$ and $|w_{n-1}| > |w_n|$. So there exists some i such that $|w_{i-1}| < |w_i| > |w_{i+1}|$. Say,

$$w_{i-1} \to w_i \text{ is an insertion of } aa^{-1}, \quad a \in \mathcal{A}$$
$$w_i \to w_{i+1} \text{ is a deletion of } bb^{-1}, \quad b \in \mathcal{A}$$

Construction of a free group 4

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- If aa^{-1} and bb^{-1} are the same letters in w_i, then we can suppress w_i and take $w_{i-1} = w_{i+1}$.
- If aa^{-1} and bb^{-1} are disjoint in w_i then we change the order: first delete bb^{-1}, then insert aa^{-1}.
- If aa^{-1} and bb^{-1} have one letter in common in w_i, for example:

$$\begin{aligned} w_{i-1} &= [...xyz...] \\ w_i &= [...xaa^{-1}yz...] \quad , \quad y = a \\ w_{i+1} &= [...xaz...] \end{aligned}$$

then we can take $w_{i-1} = w_{i+1}$. All three are decreasing $\sum |w_i|$, a contradiction. □

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Definition

The free group over X is the set $F(X)$ endowed with the product $*$ defined by: $w * w'$ is the unique reduced word equivalent to the word ww'. The unit is the empty word.

Example

1. If $\#X = 1$ then $F(X) \simeq \mathbb{Z}$.
2. IF $\#X \geq 2$ then $F(X)$ not abelian.

Terminology: We say that a free non-abelian group is a group $F(X)$ with $\mathrm{card}(X) \geq 2$.

Examples of free groups in real life: the ping-pong lemma

Example

Take $r \in \mathbb{R}$, $r \geq 2$,

$$g_1 = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}.$$

$SL(2, \mathbb{R})$ acts by isometries on $\mathbb{H}^2 = \{z \in \mathbb{C} : Im(z) > 0\}$, *via*

$$g \cdot z = \frac{az + b}{cz + d}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

Statement: We have that $\langle g_1, g_2 \rangle \leq SL(2, \mathbb{R})$ is isomorphic to $F(\{g_1, g_2\})$.

Why $\langle g_1, g_2 \rangle$ is free

Statement: We have that $\langle g_1, g_2 \rangle \leq SL(2, \mathbb{R})$ is isomorphic to $F(\{g_1, g_2\})$

Proof

$g_1(z) = z + r,\ g_2(z) = \frac{z}{rz+1}$.

$I(z) = -\frac{1}{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z$, $g_2 = I \circ g_1^{-1} \circ I^{-1}$.