PS3 Q1

(a) Van Dyke's matching rule $(m\,t\,i)(n\,t\,o) = (n\,t\,o)(m\,t\,i)$

↳ n terms of the outer solution, written in the inner variable and then expanded to m terms, is the same as m terms of the inner solution, written in terms of the outer variable and then expanded to n terms.

(b) $f(x;\varepsilon) = [1+(x+\varepsilon)^{1/2}]^{1/2}$

$\varepsilon \to 0^+$ with $x = O(1)$ $\Rightarrow$
$$f(x;\varepsilon) = [1+x^{\frac{1}{2}}(1+\varepsilon/x)^{1/2}]^{1/2}$$
$$\sim [1+x^{\frac{1}{2}}(1+\tfrac{\varepsilon}{2x}+\dots)]^{\frac{1}{2}}$$
$$= [(1+x^{\frac{1}{2}})+\tfrac{\varepsilon}{2x^{1/2}}+\dots]^{\frac{1}{2}}$$
$$= (1+x^{\frac{1}{2}})^{\frac{1}{2}}\left[1+\frac{\varepsilon}{2x^{1/2}(1+x^{1/2})}+\dots\right]^{\frac{1}{2}}$$
$$\sim (1+x^{\frac{1}{2}})^{\frac{1}{2}}\left[1+\frac{\varepsilon}{4x^{1/2}(1+x^{1/2})}+\dots\right]$$
$$= (1+x^{1/2})^{1/2}+\frac{\varepsilon}{4x^{1/2}(1+x^{1/2})^{1/2}}$$

$\therefore$ $(1\,t\,o) = (1+x^{1/2})^{1/2}$

$(2\,t\,o) = (1+x^{1/2})^{1/2} + \dfrac{\varepsilon}{4x^{1/2}(1+x^{1/2})^{1/2}}$

$\varepsilon \to 0^+$ with $X = \frac{x}{\varepsilon}$ and $X = O(1)$ $\Rightarrow$
$$f(\varepsilon X;\varepsilon) = [1+(\varepsilon X+\varepsilon)^{1/2}]^{1/2}$$
$$= [1+\varepsilon^{1/2}(X+1)^{1/2}]^{1/2}$$
$$\sim 1+\tfrac{1}{2}\varepsilon^{1/2}(X+1)^{1/2}+\dots$$

$\therefore$ $(1\,t\,i) = 1$

$(2\,t\,i) = 1+\varepsilon^{1/2}(X+1)^{1/2}$

<u>$(m,n) = (1,1)$</u>

$$(1\,t\,o) = (1+x^{1/2})^{1/2}$$
$$= (1+(\varepsilon X)^{1/2})^{1/2}$$
$$\sim 1+\tfrac{1}{2}\varepsilon^{1/2}X^{1/2}+\dots$$

$(m,n) = (1,1)$

$$(1\,t\,o) = (1+x^{1/2})^{1/2}$$
$$= (1+(\varepsilon X)^{1/2})^{1/2}$$
$$\sim 1+\tfrac{1}{2}\varepsilon^{1/2}X^{1/2}+\dots$$

$$(1\,t\,i)(1\,t\,o) = \underline{1}$$

$$(1\,t\,i) = 1$$
$$(1\,t\,o)(1\,t\,i) = \underline{1}$$

hence $(1\,t\,o)(1\,t\,i) = (1\,t\,i)(1\,t\,o)$ ✓✓

$(m,n) = (1,2)$

$$(2\,t\,o) = (1+x^{1/2})^{1/2} + \frac{1}{4x^{1/2}(1+x^{1/2})^{1/2}}$$
$$= (1+(\varepsilon X)^{1/2})^{1/2} + \frac{1}{4(\varepsilon X)^{1/2}(1+(\varepsilon X)^{1/2})^{1/2}}$$

expand

$$\sim 1+\varepsilon^{1/2}X^{1/2} + \frac{\varepsilon^{1/2}}{4X^{1/2}}$$

$$(1\,t\,i)(2\,t\,o) = 1$$

$$(1\,t\,i) = 1$$
$$\Rightarrow (2\,t\,o)(1\,t\,i) = 1$$

hence, $(1\,t\,i)(2\,t\,o) = (2\,t\,o)(1\,t\,i)$ ✓✓

$(m,n) = (2,1)$

$$(1\,t\,o) = (1+x^{1/2})^{1/2}$$
$$= (1+(\varepsilon X)^{1/2})^{1/2}$$
$$\sim 1+\tfrac{1}{2}\varepsilon^{1/2}X^{1/2}+\dots$$

$$(2\,t\,i)(1\,t\,o) = 1+\tfrac{1}{2}\varepsilon^{1/2}X^{1/2}$$

$$(2\,t\,i) = 1+\tfrac{1}{2}\varepsilon^{1/2}(X+1)^{1/2}$$
$$= 1+\tfrac{1}{2}\varepsilon^{1/2}(x/\varepsilon+1)^{1/2}$$
$$= 1+\tfrac{1}{2}x^{1/2}(1+\varepsilon/x)^{1/2}$$
$$\sim 1+\tfrac{1}{2}x^{1/2}+\dots$$

$$(1\,t\,o)(2\,t\,i) = 1+\tfrac{1}{2}x^{1/2}$$

Hence $(2\,t\,i)(1\,t\,o) = (1\,t\,o)(2\,t\,i)$ ✓✓

$(m,n) = (2,2)$

$$(2\,t\,o) = (1+x^{1/2})^{1/2} + \frac{1}{4x^{1/2}(1+x^{1/2})^{1/2}}$$
$$= (1+(\varepsilon X)^{1/2})^{1/2} + \frac{1}{4(\varepsilon X)^{1/2}(1+(\varepsilon X)^{1/2})^{1/2}}$$
$$\sim 1+\tfrac{1}{2}\varepsilon^{1/2}X^{1/2} + \frac{\varepsilon^{1/2}}{4X^{1/2}} + \dots$$
$$= 1+\varepsilon^{1/2}\left(\tfrac{1}{2}X^{1/2} + \frac{1}{4X^{1/2}}\right) + \dots$$

$$(2\,t\,i)(2\,t\,o)$$
$$= 1+\varepsilon^{1/2}\left(\tfrac{1}{2}X^{1/2} + \frac{1}{4X^{1/4}}\right)$$

$$(2ti) = 1 + \tfrac{1}{2}\varepsilon^{1/2}(X+1)^{1/2}$$
$$= 1 + \tfrac{1}{2}\varepsilon^{1/2}(x/\varepsilon + 1)^{1/2}$$
$$\sim 1 + \tfrac{1}{2}x^{1/2} + \frac{\varepsilon}{4x^{1/2}} + \cdots$$

Hence $(2ti)(2to) = (2to)(2ti)$ ✓✓.

$$(2to)(2ti) = 1 + \tfrac{1}{2}x^{1/2} + \frac{\varepsilon}{4x^{1/2}}$$

(c) $g(x) = 1 + \dfrac{\log x}{\log\varepsilon}$ with $\varepsilon \to 0^+$, $x = O(1)$ and $X = \frac{x}{\varepsilon}$ with $X \sim O(1)$.

$$g(x;\varepsilon) \sim \begin{cases} 1 + \dfrac{\log x}{\log \varepsilon} & \text{as } \varepsilon \to 0^+ \text{ with } x = O(1) \\ 2 + \dfrac{\log X}{\log\varepsilon} & \text{as } \varepsilon \to 0^+ \text{ with } X = O(1) \text{ and } X = \frac{x}{\varepsilon} \end{cases}$$

then, $(1to) = 1$ and $(1ti) = 2 \Rightarrow (1ti)(1to) = 1 \neq 2 = (1to)(1ti)$.

We can resolve the situation by treating $\log\varepsilon$ as $O(1)$ in the matching procedure:

$$(1to) = 1 + \frac{\log x}{\log\varepsilon} = 1 + \frac{\log(\varepsilon X)}{\log\varepsilon} = 2 + \frac{\log X}{\log\varepsilon} = (1ti)(1to)$$

$$(1ti) = 2 + \frac{\log X}{\log\varepsilon} = 2 + \frac{\log(x/\varepsilon)}{\log\varepsilon} = 1 + \frac{\log x}{\log\varepsilon} = (1to)(1ti)$$

These are now equal ☺

PS3 Q2

(a) $\varepsilon y' + y = x$ for $x > 0$ with $y(0) = 1$.

OUTER: $y \sim y_0 + \varepsilon y_1 + \dots$ gives $O(\varepsilon^0)$: $y_0 = x$

$O(\varepsilon^1)$: $y_0' + y_1 = 0 \Rightarrow y_1 = -1$

$\therefore\ y(x) \sim x - \varepsilon + \dots$

INNER: $y(x) = Y(X)$, $X = x/\varepsilon \sim O(1)$, and let $Y = Y_0 + \varepsilon Y_1 + \dots$

Then $\dfrac{dY}{dX} + Y = \varepsilon X$ for $X > 0$ with $Y(0) = 1$

$O(\varepsilon^0)$: $\dfrac{dY_0}{dX} + Y_0 = 0$, $Y_0(0) = 1 \Rightarrow Y_0 = e^{-X}$

$\dfrac{dY_1}{dX} + Y_1 = X$, $Y_1(0) = 0 \Rightarrow Y_1 = e^{-X} + X - 1$

$\therefore\ Y(X) \sim e^{-X} + \varepsilon(e^{-X} + X - 1) + \dots$

$$(2to) = x - \varepsilon = \varepsilon X - \varepsilon = \varepsilon(X - 1)$$

$$(2ti) = e^{-X} + \varepsilon(e^{-X} + X - 1) = e^{-x/\varepsilon} + \varepsilon\left(e^{-x/\varepsilon} + \frac{x}{\varepsilon} - 1\right)$$

$\sim x - \varepsilon +$ exponentially small terms

Hence $(2ti)(2to) = (2to)(2ti)$. ✓

[Note that the problem can be solved exactly to give $y(x) = (1+\varepsilon)e^{-x/\varepsilon} + x - \varepsilon$]

(b) $(x + \varepsilon)y' + y = 0$ for $x > 0$ with $y(0) = 1$.

OUTER: $y \sim y_0 + \varepsilon y_1 + \dots$ as $\varepsilon \to 0^+$ with $x = O(1)$

$O(\varepsilon^0)$: $x y_0' + y_0 = 0 \Rightarrow y_0 = \dfrac{A_1}{x}$ $\quad A_1 \in \mathbb{R}$.

$O(\varepsilon^1)$: $x y_1' + y_1 = -y_0' \Rightarrow y_1 = -\dfrac{A_1}{x^2} + \dfrac{A_2}{x}$ $\quad (A_2 \in \mathbb{R})$

INNER: $y(x) = Y(X)$ for $X > 0$ with $X = \dfrac{x}{\varepsilon}$ and $Y(0) = 1$

$Y(X) \sim Y_0(X) + \varepsilon Y_1(X) + \dots$ as $\varepsilon \to 0^+$ with $X = O(1)$

$O(\varepsilon^0)$: $\quad (1+X)\dfrac{dY_0}{dX} + Y_0 = 0, \quad Y_0(0) = 1 \Rightarrow Y_0 = \dfrac{1}{1+X}$

$O(\varepsilon^1)$: $\quad (1+X)\dfrac{dY_1}{dX} + Y_1 = 0, \quad Y_1(0) = 0 \Rightarrow Y_1 = 0$

matching

$$\frac{\varepsilon}{x+\varepsilon} = \frac{\varepsilon}{x}\left(\frac{1}{1+\varepsilon/x}\right) = \frac{\varepsilon}{x}\left(1 - \frac{\varepsilon}{x} + \cdots\right)$$

$$(2ti) = \frac{1}{1+X} = \frac{1}{1+x/\varepsilon} \sim \frac{\varepsilon}{x} \Rightarrow (2to)(2ti) = \frac{\varepsilon}{x}$$

$$(2to) = \frac{A_1}{x} + \varepsilon\left(-\frac{A_1}{x^2} + \frac{A_2}{x}\right)$$

$$= \frac{A_1}{\varepsilon X} + \varepsilon\left(\frac{-A_1}{\varepsilon^2 X^2} + \frac{A_2}{\varepsilon X}\right)$$

$$\sim \frac{1}{\varepsilon}\left(\frac{A_1}{X} - \frac{A_1}{X^2}\right) + \frac{A_2}{X} \Rightarrow (2ti)(2to) = \frac{A_1}{x} + \varepsilon\left(-\frac{A_1}{x^2} + \frac{A_2}{x}\right)$$

$$'' \ (2ti)(2to)$$

$$(2ti)(2to) = (2to)(2ti) \Rightarrow A_1 = 0, \quad A_2 = 1$$

$$\therefore \left.\begin{aligned} y &\sim \frac{\varepsilon}{x} \text{ for } x = O(1) \\ &\sim \frac{1}{1+X} \text{ for } X = O(1) \end{aligned}\right\} \text{ as } \varepsilon \to 0^+$$

$\varepsilon y'' + x^{\frac{1}{2}} y' + y = 0$ as $\varepsilon \to 0^+$ for $0<x<1$ with $y(0)=0$, $y(1)=1$.

(a) Let $x = 1 + \delta(\varepsilon) X$, $y = Y(X)$ with $X = O(1)$ and $\delta \to 0$ as $\varepsilon \to 0^+$

$$\Rightarrow \underset{①}{\frac{\varepsilon}{\delta^2} Y''} + \underset{②}{\frac{(1+\delta X)^{\frac{1}{2}}}{\delta} Y'} + \underset{③}{Y} = 0$$

balance by setting $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \Rightarrow \varepsilon = \delta$

$$\therefore \quad Y'' + (1+\varepsilon X)^{\frac{1}{2}} Y' + \varepsilon Y = 0 \quad \text{for } X<0 \text{ with } Y(0)=1$$

Expand: $Y \sim Y_0 + \varepsilon Y_1 + \dots$ as $\varepsilon \to 0^+$ with $X = O(1)$.

$O(\varepsilon^0)$: $\frac{d^2 Y_0}{dX^2} + \frac{dY_0}{dX} = 0$, $Y_0(0) = 1 \Rightarrow Y_0 = A + (1-A)e^{-X}$ $(A \in \mathbb{R})$

Then, matching as we move towards the outer solution with require $Y_0(-\infty)$ to be finite – but this can only be achieved for $A = 1$

$\Rightarrow Y_0 \equiv 1$ ie there is no boundary layer.

(b) Let $y \sim y_0 + \varepsilon y_1 + \dots$ as $\varepsilon \to 0^+$ (OUTER) with $x = O(1)$.

$O(\varepsilon^0)$: $x^{\frac{1}{2}} y_0' + y_0 = 0$, $y_0(1) = 1$

$$\Rightarrow \frac{y_0'}{y_0} = -\frac{1}{x^{\frac{1}{2}}} \Rightarrow \ln|y_0| = -2x^{\frac{1}{2}} + c$$

$$y_0 = e^{-2x^{1/2} + c}$$

$$y_0(1) = 1 \Rightarrow 1 = e^{-2+c} \Rightarrow c = e^2 \quad \therefore y_0 = e^{2(1-x^{1/2})}.$$

(c) Let $x = \delta(\varepsilon) X$, $y = Y(X)$ with $X = O(1)$, $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

$$\Rightarrow \underset{①}{\frac{\varepsilon}{\delta^2} \frac{d^2 Y}{dX^2}} + \underset{②}{\frac{(\delta X)^{1/2}}{\delta} \frac{dY}{dX}} + \underset{③}{Y} = 0$$

Hence dominant balance is ② ~ ③

NB ③ << ②

Set $\frac{\varepsilon}{\delta^2} = \frac{1}{\delta^{1/2}} \Rightarrow \delta = \varepsilon^{2/3} \Rightarrow$ BL of thickness $O(\varepsilon^{2/3})$.

(d) $\therefore \quad \frac{d^2Y}{dX^2} + X^{\frac{1}{2}}\frac{dY}{dX} + \varepsilon^{\frac{1}{3}} Y = 0$ for $X>0$ with $Y(0)=0$.

NB Scaling of the BL $\Rightarrow$ we should have expanded as $y \sim y_0 + \varepsilon^{\frac{1}{3}} y_1 + \dots$ in the outer region.

Expand: $Y \sim Y_0 + \varepsilon^{\frac{1}{3}} Y_1 + \dots$ as $\varepsilon \to 0^+$ with $X = O(1)$.

$O(\varepsilon^0)$: $\quad \frac{d^2Y_0}{dX^2} + X^{\frac{1}{2}}\frac{dY_0}{dX} = 0, \quad Y_0(0) = 0$

$$\Rightarrow \quad \frac{dY_0}{dX} = Ce^{-\frac{2}{3}X^{3/2}} \quad (C \in \mathbb{R})$$

$$\therefore \quad Y_0 = C\int_0^X e^{-\frac{2}{3}t^{3/2}}\,dt$$

Matching:

$$(1 \text{ t } o) = e^{2(1-x^{1/2})} = e^{2(1-(\varepsilon^{2/3}X)^{1/2})} = e^2 e^{-2\varepsilon^{1/3}X^{1/2}} \sim e^2$$

$$(1 \text{ t } i) = C_0\int_0^x e^{-\frac{2}{3}t^{3/2}}\,dt = C_0\int_0^{x/\varepsilon^{2/3}} e^{-\frac{2}{3}t^{3/2}}\,dt \sim C_0\int_0^\infty e^{-\frac{2}{3}t^{3/2}}\,dt$$

$\therefore (1 \text{ t } i)(1 \text{ t } o) = e^2$

$\therefore (1 \text{ t } o)(1 \text{ t } i) = C_0\int_0^\infty e^{-\frac{2}{3}t^{3/2}}\,dt$

Hence, by VDMR

$$e^2 = C_0\int_0^\infty e^{-\frac{2}{3}t^{3/2}}\,dt = C_0\left(\frac{2}{3}\right)^{1/3}\int_0^\infty s^{2/3-1}e^{-s}\,ds = C_0\left(\frac{2}{3}\right)^{1/3}\Gamma\left(\frac{2}{3}\right)$$

$s = \frac{2}{3}t^{3/2} \Rightarrow \frac{ds}{dt} = t^{1/2}$

$$\therefore \quad C_0 = \frac{e^2}{\left(\frac{2}{3}\right)^{1/3}\Gamma\left(\frac{2}{3}\right)}$$

(a) $\varepsilon y'' + yy' - y = 0$ $\quad 0<x<1$ with $y(0)=1,\ y(1)=3$ $\quad$ as $\varepsilon \to 0^+$

OUTER: $y \sim y_0 + \varepsilon y_1 + \dots$ as $\varepsilon \to 0^+$ with $x = O(1)$.

$O(\varepsilon^0)$: $\quad y_0' y_0 - y_0 = 0 \quad 0<x<1$ and $y_0(1) = 3$ (since no BL @ RH end)

$\therefore\ y_0 = x+2$

INNER: $x = \delta(\varepsilon) X,\ y = Y(X)$ with $\delta(\varepsilon) \to 0^+,\ X = O(1)$ as $\varepsilon \to 0^+$.

$$\Rightarrow \underset{①}{\frac{\varepsilon}{\delta^2}\frac{d^2Y}{dX^2}} + \underset{②}{\frac{1}{\delta} Y \frac{dY}{dX}} - \underset{③}{Y} = 0$$

③ ≪ ②

dominant balance ① ~ ②

ie $\dfrac{\varepsilon}{\delta^2} = \dfrac{1}{\delta} \Rightarrow \delta = \varepsilon$.

Expand $Y \sim Y_0 + \varepsilon Y_1 + \dots$ as $\varepsilon \to 0^+$ with $X = O(1)$

$O(\varepsilon^0)$: $\quad \dfrac{d^2 Y_0}{dX^2} + Y_0 \dfrac{dY_0}{dX} = 0 \Rightarrow \dfrac{dY_0}{dX} + \frac{1}{2} Y_0^2 = \frac{1}{2} B_1 \quad (B_1 \in \mathbb{R})$

with $Y_0(0) = 1$.

Let $B_1 = -w^2$ $(w>0)$. Then $Y_0 = w \tan\left(\frac{w}{2}(X_0 - X)\right)$

and $Y_0(0) = 1 \Rightarrow 1 = w\tan\left(\frac{w}{2}X_0\right)$ for $X_0 \in \mathbb{R}$ ← Cannot match since will get a singularity in $Y_0(X)$.

Let $B_1 = 0 \Rightarrow Y_0 = \dfrac{1}{1+X/2} \Rightarrow$ Cannot match as $Y_0(\infty) = 1 \neq 2 = y_0(0^+)$

Let $B_1 = w^2$ $(w>0)$. Then $Y_0 = w\tanh\left(\frac{w}{2}(X - X_0)\right)$

and $Y_0(0) = 1 \Rightarrow 1 = w\tanh\left(-\frac{w}{2}X_0\right)$ for $X_0 \in \mathbb{R}$.

and $Y_0(\infty) = w \Rightarrow$ Can match with the outer.

So, for a valid solution we need $B_1 = w^2 > 0$

Matching

$(1t0) = x+2$
$\quad = \varepsilon X + 2$

$(1ti)(1t0) = 2$

$(1ti) = w\tanh\left(\frac{w}{2}(X - X_0)\right)$
$\quad = w\tanh\left(\frac{w}{2}\left(\frac{x}{\varepsilon} - X_0\right)\right)$
$\quad \sim w$ as $\varepsilon \to 0^+$ $\quad w>0,\ x>0$

$\Rightarrow (1t0)(1ti) = w$ and $X_0 = -\tanh^{-1}\left(\frac{1}{2}\right)$

(b) $\varepsilon y'' + yy' - y = 0 \quad 0<x<1$ with $y(0) = -\frac{3}{4}$ and $y(1) = \frac{5}{4}$ as $\varepsilon \to 0^+$

LH OUTER: $y \sim y_{L0} + \varepsilon y_{L1} + \cdots$ as $\varepsilon \to 0^+$ with $0 < x < x_0$

$O(\varepsilon^0)$: $y_{L0}' y_{L0} - y_{L0} = 0$ with $y_{L0}(0) = -\frac{3}{4} \Rightarrow y_{L0} = x - \frac{3}{4}$

$0 < x < x_0$

RH OUTER: $y \sim y_{R0} + \varepsilon y_{R1} + \cdots$ as $\varepsilon \to 0^+$ with $x_0 < x < 1$

$O(\varepsilon^0)$: $y_{R0}' y_{R0} - y_{R0} = 0$ with $y_{R0}(0) = \frac{5}{4} \Rightarrow y_{R0} = x + \frac{1}{4}$

$x_0 < x < 1.$

INNER: $x = x_0 + \varepsilon X$, $y = Y(X) \sim Y_0(X)$ as $\varepsilon \to 0^+$ with $X \sim O(1)$.

$O(\varepsilon^0)$: $\dfrac{d^2 Y_0}{dX^2} + Y_0 \dfrac{dY_0}{dX} = 0$ for $-\infty < X < \infty$

$$\Rightarrow \frac{dY_0}{dX} + \frac{1}{2} Y_0^2 = \frac{1}{2} w^2 > 0$$ (to avoid a singularity at finite X, as per (a))

$$\therefore \; Y_0(X) = w \left(\frac{Be^{wX} - 1}{Be^{wX} + 1} \right) \quad (B \in \mathbb{R})$$

Matching: $y_{L0}(x_0^-) = Y_0(-\infty)$ and $y_{R0}(x_0^+) = Y_0(+\infty)$

$x_0 - \frac{3}{4} = -w \qquad\qquad x_0 + \frac{1}{4} = w$

$$\Rightarrow w = \frac{1}{2} \text{ and } x_0 = \frac{1}{4}.$$

[Note that the constant B is still undetermined. This will be the case $\forall$ the IN$_0$! We would need a WKB analysis to pin it down.]

PS3 Q5

$y'' + \varepsilon y' = 0$ as $\varepsilon \to 0^+$ with $0 < x < L$ and $y(0) = 0$, $y(L) = 1$.

(a) Suppose $L = O(1)$ as $\varepsilon \to 0^+$. Let $y = y_0 + \varepsilon y_1 + \dots$ as $\varepsilon \to 0^+$

$O(\varepsilon^0)$: $y_0'' = 0$ with $y_0(0) = 0$, $y_0(L) = 1 \Rightarrow y_0 = \frac{x}{L}$.

$O(\varepsilon^1)$: $y_1'' + y_0' = 0$ for $0 < x < 1$ with $y_1(0) = 0$, $y_1(L) = 0$

$\Rightarrow y_1'' = -\frac{1}{L}$ $\quad \therefore \quad y_1 = \frac{1}{2L} x(L - x)$

$\therefore \; y(x) \sim \frac{x}{L} + \varepsilon \cdot \frac{1}{2L} x(L-x) + \dots$ as $\varepsilon \to 0^+$ with $L = O(1)$.

(b) Note that the expansion is not valid for $L \gg \frac{1}{\varepsilon}$ (and hence in the large L ($L \to \infty$) limit).

Differentiating gives $y'(x) \sim \frac{1}{L} + \frac{\varepsilon}{2L}(L - 2x) + \dots$ as $\varepsilon \to 0^+$ with $L = O(1)$

$\Rightarrow \; y'(0) \sim \frac{1}{L} + \frac{\varepsilon}{2} + \dots$ as $\varepsilon \to 0^+$

So this expansion is not valid when $\frac{\varepsilon}{L} = o(1)$ as $\varepsilon \to 0^+$. This corresponds to a distinguished limit in which we have $L = \frac{\ell}{\varepsilon}$ with $\ell = O(1)$ as $\varepsilon \to 0^+$.

Scaling $x = \frac{X}{\varepsilon}$ and $y = Y(X) \Rightarrow Y'' + Y' = 0$ for $0 < X < \ell$

with $Y(0) = 0$, $Y(\ell) = 1$. Hence $Y(X) = \dfrac{1 - e^{-X}}{1 - e^{-L}}$.

In this case (we have scaled properly) we have

$Y'(0) = \frac{1}{1 - e^{-\ell}} \to 1$ as $\ell \to \infty$. This agrees with what is obtained from the exact soln $y = \dfrac{1 - e^{-\varepsilon x}}{1 - e^{-\varepsilon L}} \Rightarrow y'(0) = \varepsilon$ as $L \to \infty$.

1a) $\varepsilon \nabla^2 u = u$ in $r^2 = x^2 + y^2 < 1$ with $u = 1$ on $r = 1$ as $\varepsilon \to 0^+$

OUTER: $u \sim u_0 + \varepsilon u_1 + \dots$ as $\varepsilon \to 0^+$ with $1 - r \sim O(1)$

$$\left.\begin{aligned} O(\varepsilon^0) &: \quad u_0 = 0 \\ O(\varepsilon^1) &: \quad u_1 = \nabla^2 u_0 \quad \Rightarrow \quad u_1 = 0 \\ O(\varepsilon^2) &: \quad u_2 = \nabla^2 u_1 \quad \Rightarrow \quad u_2 = 0 \end{aligned}\right\} \quad u = o(\varepsilon^n) \ \forall n \in \mathbb{N} \text{ as } \varepsilon \to 0^+.$$

INNER: $u(r,\theta) = U(R,\theta)$ with $r = 1 - \delta(\varepsilon) R$ with $\delta(\varepsilon) \to 0$ and $R = O(1)$ as $\varepsilon \to 0^+$.

$$\Rightarrow \quad \underbrace{\frac{\varepsilon}{\delta^2} U_{RR}}_{1} - \underbrace{\frac{\varepsilon}{\delta(1-\delta R)} U_R}_{2} + \underbrace{\frac{\varepsilon}{(1-\delta R)^2} U_{\theta\theta}}_{3} - \underbrace{U}_{4} = 0$$

balance by setting $\delta = \varepsilon^{\frac{1}{2}}$

$$\Rightarrow \quad U_{RR} - \frac{\varepsilon^{\frac{1}{2}}}{(1 - \varepsilon^{\frac{1}{2}} R)} U_R + \frac{\varepsilon}{(1 - \varepsilon^{1/2} R)^2} U_{\theta\theta} - U = 0$$

Expand: $U \sim U_0(R,\theta) + \varepsilon^{\frac{1}{2}} U_1(R,\theta) + \dots$ as $\varepsilon \to 0^+$ with $R = O(1)$.

$O(\varepsilon^0)$: $U_{0_{RR}} - U_0 = 0$ in $R > 0$ with $U_0 = 1$ on $R = 0$

$$\Rightarrow \quad U_0 = A e^{R} + (1 - A) e^{-R} \qquad (A \in \mathbb{R})$$

Matching: $(1t0) = 0 \quad \Rightarrow \quad (1ti)(1t0) = 0$

$\Rightarrow \quad (1t0)(1ti) = 0$ (by VDMR)

$\Rightarrow \quad U_0 \to 0$ as $R \to \infty$

$\Rightarrow \quad A = 0$

$\therefore \quad u = e^{-R} + O(\varepsilon^{1/2})$ as $\varepsilon \to 0^+$ with $\varepsilon^{1/2}(1-r) = R = O(1)$.

Exact solution: $u = \dfrac{I_0(r/\sqrt{\varepsilon})}{I_0(1/\sqrt{\varepsilon})}$

$$I_0(x) = \frac{1}{\pi}\int_0^\pi \cos(ix\sin\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^\pi \left(e^{-i(ix\sin\theta)} + e^{+i(ix\sin\theta)}\right) d\theta$$

$$= \frac{1}{2\pi}\int_0^\pi \left(e^{x\sin\theta} + e^{-x\sin\theta}\right) d\theta$$

$$\sim \frac{1}{2\pi}\int_0^\pi e^{x\sin\theta}\,d\theta \quad \text{as } x\to\infty$$ (1st term dominates because $\sin\theta > 0$ on $(0,\pi)$)

$$\sim \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{x[1-\frac{1}{2}(\theta-\pi/2)^2+\dots]}\,d\theta$$ (use Laplace's method because $\phi(\theta)=\sin\theta$ has a maximum at $\theta=\frac{\pi}{2}$)

$$\sim \frac{e^x}{2\pi}\int_{-\infty}^{\infty} e^{-xs^2/2}\,ds$$ $\leftarrow$ $(\theta-\pi/2 = s)$

$$= \frac{e^x}{\sqrt{2\pi}}\sqrt{\frac{2}{x}}\underbrace{\int_{-\infty}^{\infty} e^{-t^2}\,dt}_{=\sqrt{\pi}}$$ $\leftarrow$ $\left(s=\sqrt{\tfrac{2}{x}}\,t\right)$

$$\therefore\ I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as } x\to\infty.$$

Hence $u \sim \dfrac{1}{\sqrt{r}}e^{-(1-r)/\sqrt{\varepsilon}}$ as $\varepsilon\to 0^+$ with $r=O(1)$, $1-r=O(1)$

$u \sim \dfrac{\sqrt{2\pi}\,e^{-1/\sqrt{\varepsilon}}}{\varepsilon^{1/4}}\, I_0(\rho)$ as $\varepsilon\to 0^+$ with $\rho = \varepsilon^{-\frac{1}{2}}r = O(1)$

$r = 1-\varepsilon^{1/2}R \Rightarrow 1-r = \varepsilon^{1/2}R$

$u \sim \dfrac{1}{\sqrt{1-\varepsilon^{1/2}R}}\,e^{-R} = e^{-R} + O(\varepsilon^{\frac{1}{2}})$ as $\varepsilon\to 0^+$ with $R = \varepsilon^{-\frac{1}{2}}(1-r) = O(1)$

$\Rightarrow$ consistent with the result from BL expansion.

(6) $\varepsilon\nabla^2 u = u_x$ in $y>0$ with $u=1$ on $y=0$, $x>0$

$u_y = 0$ on $y=0$, $x<0$

$u \to 0$ as $x^2+y^2 \to \infty$, $y>0$

OUTER: $u \sim u_0 + \varepsilon u_1 + \dots$ as $\varepsilon \to 0^+$ with $x, y = O(1)$.

$O(\varepsilon^0)$: $u_{0x} = 0$ with $u_0 = 0$ at ∞ $\Rightarrow$ $u_0 \equiv 0$.

$O(\varepsilon^1)$: $u_{1x} = 0$ with $u_1 = 0$ at ∞ $\Rightarrow$ $u_1 \equiv 0$

$u = o(\varepsilon^n)$ $\forall n \in \mathbb{N}$ as $\varepsilon \to 0^+$ with $x, y = O(1)$.

INNER: $u(x,y) = U(x,Y)$ with $y = \delta(\varepsilon) Y$ and $\delta \to 0$, $Y = O(1)$ as $\varepsilon \to 0^+$

$$\Rightarrow \varepsilon U_{xx} + \underbrace{\frac{\varepsilon}{\delta^2} U_{YY} - U_x}_{\text{Balance}} = 0$$

Balance $\Rightarrow$ $\delta = \varepsilon^{\frac{1}{2}}$ $\Rightarrow$ $\varepsilon U_{xx} + U_{YY} - U_x = 0$.

$U \sim U_0 + \varepsilon U_1 + \dots$ as $\varepsilon \to 0^+$ with $Y = O(1)$.

$O(\varepsilon^0)$: $U_{0YY} - U_{0x} = 0$ in $Y>0$, $x>0$ with $U_0(x,0) = 1$ for $x>0$

Matching: $(1 t o) = 0$ $\Rightarrow$ $(1 t i)(1 t o) = 0$

$\Rightarrow$ $(1 t o)(1 t i) = 0$ $\quad$ ↓ VDMR

$\Rightarrow$ $U_0 \to 0$ as $Y \to \infty$ for $x>0$

Seek a similarity solution $U_0 = f(\eta)$ with $\eta = Y/\sqrt{x}$.

Substituting: $\eta_x = -\dfrac{\eta}{2x}$, $\eta_Y = \dfrac{1}{x^{1/2}}$

$$\therefore \quad U_{0x} = f'(\eta)\,\eta_x = -\frac{\eta f'(\eta)}{2x}$$

$$U_{0YY} = f''(\eta)\,\eta_Y^2 = \frac{f''(\eta)}{x}$$

$$\Rightarrow f'' + \tfrac{1}{2}\eta f' = 0 \quad (\eta > 0)$$

BCs $U_0 = 1$ on $Y=0$, $x>0$ $\Rightarrow$ $f(0) = 1$

$U_0 \to 0$ as $Y \to \infty$, $x>0$ $\Rightarrow$ $f(\infty) = 0$

$$\therefore \quad \frac{f''(\eta)}{f'(\eta)} = -\tfrac{1}{2}\eta \quad \Rightarrow \quad \ln|f'(\eta)| = c_1 - \tfrac{1}{4}\eta^2 \qquad (c_1 \in \mathbb{R})$$

$$\therefore \quad f'(\eta) = e^{c_1 - \frac{1}{4}\eta^2}$$

$$f(\eta) = c_2 - c_1 \int_\eta^\infty e^{-\frac{1}{4}s^2}\, ds$$

$s = 2t$

$$= c_2 - 2c_1 \int_{\eta/2}^\infty e^{-t^2}\, dt$$

$$= c_2 - 2c_1 \operatorname{erfc}(\eta/2)$$

$$\left.\begin{aligned} f(\infty) = 0 &\Rightarrow c_2 = 0 \\ f(0) = 1 &\Rightarrow 1 = -2c_1 \operatorname{erfc}(0) = -2c_2 \end{aligned}\right\} \Rightarrow f(\eta) = \operatorname{erfc}(\eta/2)$$

$\therefore\ u = \operatorname{erfc}\left(\frac{y}{2\sqrt{x}}\right) + o(\varepsilon)$ as $\varepsilon \to 0^+$ with $Y = \varepsilon^{-\frac{1}{2}} y = O(1)$
and $x = O(1)$

Neither approximation holds for $X = \frac{x}{\varepsilon} = O(1)$, $Y = \frac{y}{\varepsilon} = O(1)$

$\Rightarrow\ u_{XX} + u_{YY} = u_X$ in $Y > 0$.