C3.3 Differentiable manifolds – Class 4 (22 Jan 2025)

Part B.

5. Let X^k be a compact oriented manifold and $f: X \to Z$ a smooth map. Let $\alpha \in \Omega^k(Z)$ be a closed k-form on Z.

(i) Show by integrating $f^*(\alpha)$ on X that f defines a linear map $L_f: H^k(Z) \to \mathbb{R}$.

proof. Pullbacks and integration are clearly linear. If $\alpha = d\beta$,

$$\int_X f^*(d\beta) = \int_X d(f^*\beta) = 0$$

by compatibility of the pullbacks and by Stokes's theorem. Therefore

$$L_f(\alpha) = \int_X f^*(\alpha)$$ is well-defined $H^k(Z) \to \mathbb{R}$. □

(ii) Let Y^{k+1} be a compact oriented manifold with boundary, $\partial Y = X$ and $g|_{\partial Y} = f$. Show that $L_f = 0$.

proof.

$$L_f(\alpha) = \int_X f^*(\alpha) = \int_{\partial Y} g^*(\alpha) = \int_Y d\, g^*(\alpha)$$

$$= \int_Y g^*(d\alpha) = 0 \quad \text{since } d\alpha = 0. \quad □$$

6 (a). On $S^1 \subset \mathbb{R}^2$ let $d\theta = \frac{dx_2}{x_1} = \frac{-dx_1}{x_2}$.

Consider the n-torus $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$ and let

$\pi_i : T^n \to S^1$ be the projection onto the ith factor.

Show that $\{\pi_i^*([d\theta])\}$, $i = 1, \dots, n$, are linearly independent in $H^1(T^n)$.

proof. Take coordinate θ_i on the i th S^1, and coords. $(\theta_1, \dots, \theta_n)$ on T^n. $\theta_j = e^{i\theta_j}$. Then

$$\pi_1^*(d\theta) \wedge \dots \wedge \pi_n^*(d\theta) = d\theta_1 \wedge \dots \wedge d\theta_n$$

which is nonvanishing & defines an orientation.

We see that the cohomology class in $H^n(T^n)$ of $d\theta_1 \wedge \dots \wedge d\theta_n$ must be nonzero, since $\int_{T^n} d\theta_1 \wedge \dots \wedge d\theta_n = (2\pi)^n$ and if we assumed for sake of contradiction that

$d\theta_1 \wedge \dots \wedge d\theta_n = d\beta$, then "$\int_{T^n} d\beta = \int_{\partial T^n} \beta = 0$ since T^n has no boundary"; this is a contradiction.

Recalling the properties of the cup product and that $[\pi_i^*(d\theta)] = \pi_i^*[d\theta]$, we get that

$$\pi_1^*[d\theta] \cup \dots \cup \pi_n^*[d\theta] \text{ is nonzero.}$$

Clearly, if $\pi_j^*[d\theta]$ is a liner combination of the others then the above cup product is zero by the antisymmetry of the cup product on degree 1 classes, which is a contradiction.

(b) Let $n>1$ and $f: S^n \to T^n$ smooth. Show that $\deg f = 0$.

<u>proof</u>. We have

$$f^*\left(\pi_1^*[d\theta] \cup \dots \cup \pi_n^*[d\theta]\right)$$

$$= f^*(\pi_1^*[d\theta]) \cup \dots \cup f^*(\pi_n^*[d\theta])$$

but $f^*(\pi_j^*[d\theta]) \in H^1(S^n) = 0$

so the whole thing is $=0$. However

$\pi_1^*[d\theta] \cup \dots \cup \pi_n^*[d\theta] \neq 0$ by (a) and $f^*: H^n(T^n) \to H^n(S^n)$

is multiplication by $\deg f$ $\mathbb{Z} \to \mathbb{Z}$, so $\deg f = 0$. $\square$

7. Consider the quaternions

$$\mathbb{H} = \{ q = x_0 + ix_1 + jx_2 + kx_3 \mid i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j \}.$$

(a) Show that if $\bar{q} = x_0 - ix_1 - jx_2 - kx_3$ then $q\bar{q} = \|q\|^2$ and $\|ab\|^2 = \|a\|^2 \|b\|^2$.

<u>proof</u>. $q\bar{q} = (x_0 + ix_1 + jx_2 + kx_3)(x_0 - ix_1 - jx_2 - kx_3)$

$$= x_0^2 - (ix_1 + jx_2 + kx_3)^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 = \|q\|^2$$

since $ij = -ji$, etc.

$\|ab\|^2 = ab\,\overline{ab} = ab\,\bar{b}\,\bar{a}$ (since $ij = -ji$ etc, so commutation is the same as conjugating)

$= a\|b\|^2\bar{a}$

$= \|a\|^2\|b\|^2$.

(ii) Show that $f(q) = q^2$ is a smooth map $\mathbb{R}^4 \cup \{\infty\} \cong S^4 \to S^4$.
How many solutions are there to $q^2 = 1$?

proof. Clearly this is a polynomial and consequently smooth on $\mathbb{R}^4$. Near ∞, consider the chart

$$\varphi : \mathbb{R}^4 \to S^4$$
$$p \mapsto \frac{1}{\bar{p}}.$$
$$0 \mapsto \infty$$

Its inverse is $\varphi^{-1}(q) = \frac{1}{\bar{q}}$, $\varphi^{-1}(\infty) = 0$.

Then we compute $(\varphi^{-1} \circ f \circ \varphi)(p) = \varphi^{-1}(f(\frac{1}{\bar{p}})) = \varphi^{-1}((\frac{1}{\bar{p}})^2)$

$$= \varphi^{-1}\left(\frac{p^2}{\|p\|^2}\right) = \frac{\|p\|^2}{\bar{p}^2} = p^2,$$ so f is smooth near ∞.

If $q^2 = 1$, just write $1 = (x_0 + ix_1 + jx_2 + kx_3)^2$

$$1 = x_0^2 - x_1^2 - x_2^2 - x_3^2 + 2x_0x_1 i + 2x_0x_2 j + 2x_0x_3 k$$

(since $ij = -ji$ etc.)

Case 1: $x_0 = 0$. This is clearly impossible.

Case 2: $x_1 = x_2 = x_3 = 0$. Then $1 = x_0^2$, so $x_0 = \pm 1$ are the 2 solutions..

(iii) What is the degree of f?

We compute $T_q f$ at $q = \pm 1$:

$$f(q) = x_0^2 - x_1^2 - x_2^2 - x_3^2 + 2x_0x_1 i + 2x_0x_2 j + 2x_0x_3 k$$

$$Df = \langle 2x_0 + 2x_1 i + 2x_2 j + 2x_3 k,\ -2x_1 + 2x_0 i,\ -2x_2 + 2x_0 j,\ -2x_3 + 2x_3 k \rangle.$$

at $q = \pm 1$ this is $\langle \pm 2, \pm 2i, \pm 2j, \pm 2k \rangle$, so as a result

$T_q f(\dot{q}) = \pm 2\dot{q}$.

Therefore 1 is a regular value with inverse images ± 1.
$\vec{x} \mapsto \pm 2\vec{x}$ is orientation-preserving in 4 dimensions, so the degree of f is 2.

(iv) How many solutions are there to $q^2 = -1$?

$$-1 = x_0^2 - x_1^2 - x_2^2 - x_3^2 + 2x_0x_1 i + 2x_0x_2 j + 2x_0x_3 k.$$

Case 1: $x_0 = 0$. Then $1 = x_1^2 + x_2^2 + x_3^2$ so there is a 2-sphere of solutions.

Case 2: $x_0 \neq 0$. Then $x_1 = x_2 = x_3 = 0$, $-1 = x_0^2$, which has no solutions.

8. Write in coordinates $x_2, \dots, x_n$, $(x_1 \neq 0)$ the induced metric on $S^{n-1} \in \mathbb{R}^n$. Show that its volume form is $\omega = x_1^{-1} dx_2 \wedge \dots \wedge dx_n$.

<u>Solution</u>. We can write down

$$x_1^2 + \dots + x_n^2 = 1, \quad \text{so} \quad x_1 dx_1 + \dots + x_n dx_1 = 0,$$

$$g_{Eucl} = \sum_{i=1}^n dx_i^2 = \sum_{i=2}^n dx_i^2 + \frac{1}{x_1^2}\left(\sum_{i=2}^n x_i dx_i\right)^2$$

$$= \sum_{i,j=2}^n g_{ij}\, dx_i\, dx_j, \text{ where}$$

$$g_{ij} = \delta_{ij} + \frac{1}{x_1^2} x_i x_j.$$

Take a coordinate $dy = \dfrac{\sum_{j=2} x_j dx_j}{\sqrt{\sum_{j=2}^n x_j^2}}$; then, since $y = \dfrac{(x_2, \dots, x_n)}{\sqrt{\sum_{j=2}^n x_j^2}}$ is a unit vector we can write

$$g_{Eucl}\big|_{S^{n-1}} = \delta_{ij}\, dx_i dx_j + \frac{\sum_{j=2}^n x_j^2}{x_1^2} dy^2, \text{ which is a}$$

diagonal metric with determinant $1 + \frac{\sum_{j=2}^{n} x_j^2}{x_1^2} \times 1 \cdot 1 \cdots$

$= 1 + \frac{1 - x_1^2}{x_1^2} = \frac{1}{x_1^2}$. y is a unit vector, so the change of coordinates has unit determinant & $\det g_{ij} = \frac{1}{x_1^2}$;

consequently the volume form is $\sqrt{\det g}\, dx_2 \wedge \cdots \wedge dx_n$

$= \frac{1}{x_1} dx_2 \wedge \cdots \wedge dx_n$.