Examples of null-Lagrangians

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Notations

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. We write u as a column vector

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

The gradient ∇u is a matrix-valued map from Ω into $\mathbb{R}^{n\times n}$:

$$\nabla u = \begin{pmatrix} \partial_1 u_1 & \dots & \partial_n u_1 \\ \vdots & \ddots & \vdots \\ \partial_1 u_n & \dots & \partial_n u_n \end{pmatrix}.$$

In particular, the (i,j)-entry of ∇u is $\partial_j u_i$.

Null-Lagrangian

Recall that an expression $L(\nabla u)$ where L is a function defined on $\mathbb{R}^{n\times n}$ is a null-Lagrangian if

$$\sum_j \partial_j \Big[\frac{\partial L}{\partial p_{ij}}(\nabla u)\Big] = 0 \text{ in } \Omega$$

for any smooth function $u \in C^\infty(\bar\Omega)$ and $i = 1, \dots, n$. Here p is a dummy variable for a matrix, and p_{ij} are its entries.

σ_k-functions of the gradient are null-Lagrangians

Define the σ_k functions on $\mathbb{R}^{n\times n}$ by

$$\det(\lambda I - p) = \lambda^n - \sigma_1(p)\,\lambda^{n-1} + \dots + (-1)^n\,\sigma_n(p)\,I$$

Note that $\sigma_0(p) = I$, $\sigma_1(p) = \operatorname{tr} p$, $\sigma_n(p) = \det p$.

Proposition 1. *$\sigma_k(\nabla u)$ is a null-Lagrangian for $0 \le k \le n$.*

Proof. Define the Newton tensor $T_k(p)$ of a matrix p by

$$T_k(p) = p^k - \sigma_1(p)\, p^{k-1} + \ldots + (-1)^k \sigma_k(p)\, I.$$

Note that $T_0(0) = I$ and $T_n(p) = 0$. It turns out that the first derivative of σ_k is given by the Newton tensor:

$$\frac{\partial \sigma_k}{\partial p_{ij}}(p) = (-1)^{k-1}(T_{k-1}(p))_{ji}.$$

(For extra fun, prove this algebraic fact.)

Thus, we only need to show that, for any $u \in C^\infty(\bar{\Omega})$ and $0 \le k \le n-1$,

$$\sum_j \partial_j \Big[(T_k(\nabla u))_{ji}\Big] = 0 \text{ in } \Omega. \tag{*}$$

Indeed, observe that

$$T_k(p) = T_{k-1}(p)p + (-1)^k \sigma_k(p)\, I. \tag{**}$$

This identity allows us to prove (*) by induction. When $k = 0$, (*) is clear as $T_0 = I$. Suppose (*) holds for some k. Then, by (**),

$$\begin{aligned}
(T_{k+1}(\nabla u))_{ji} &= \sum_l (T_k(\nabla u))_{jl} (\nabla u)_{li} + (-1)^{k+1} \sigma_{k+1}(\nabla u)\, \delta_{ij} \\
&= \sum_l (T_k(\nabla u))_{jl} \partial_i u_l + (-1)^{k+1} \sigma_{k+1}(\nabla u)\, \delta_{ij},
\end{aligned}$$

and so

$$\begin{aligned}
\sum_j \partial_j \Big[(T_{k+1}(\nabla u))_{ji}\Big] &= \sum_{j,l} (T_k(\nabla u))_{jl} \partial_j \partial_i u_l + \sum_l \partial_i u_l \sum_j \partial_j \Big[(T_k(\nabla u))_{li}\Big] \\
&\quad + (-1)^{k+1} \partial_i \sigma_{k+1}(\nabla u).
\end{aligned}$$

Note that the middle term vanishes due to induction hypothesis. Also, by the chain rule

$$\partial_i \sigma_{k+1}(\nabla u) = \sum_{r,s} \frac{\partial \sigma_{k+1}}{\partial p_{rs}}(\nabla u) \partial_i (\nabla u)_{rs} = (-1)^k \sum_{r,s} (T_k(\nabla u))_{sr} \partial_i \partial_s u_r$$

We thus have

$$\sum_j \partial_j \Big[(T_{k+1}(\nabla u))_{ji}\Big] = \sum_{j,l} (T_k(\nabla u))_{jl} \partial_j \partial_i u_l - \sum_{r,s} (T_k(\nabla u))_{sr} \partial_i \partial_s u_r$$

Relabeling r as l and s as j in the second sum, we see that this cancels out completely. We have thus shown (*) with k replaced by $k+1$. $\square$