Geometric Group Theory

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Inspirational quotations

Henri Poincaré: "If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living."

Jean-le-Rond D'Alembert, to his students (quoted by Florian Cajori in "A history of mathematics"): "Allez en avant et la foi vous viendra."

"Keep going and faith will come later."

Algorithmic problems for infinite groups

Proposition

1. *Every $u \in F(X)$ is conjugate to a cyclically reduced word.*
2. *If $u, v \in F(X)$ are cyclically reduced then they are conjugate if and only if they are cyclic permutations of each other.*

Corollary

All non-trivial elements in $F(X)$ have infinite order.

Proof.

For all non-trivial $w \in F(X)$, $w = gug^{-1}$, u cyclically reduced and non-trivial. And for all cyclically reduced non-trivial u, u^n is reduced and hence $\neq w_\emptyset$.

Thus, u has infinite order, therefore w has infinite order. □

Corollary (unique root property)

If $g, h \in F(X)$ are such that $g^k = h^k$ for some k then $g = h$.

Question: Find a torsion-free group G for which there exist $g \neq h$ such that $g^k = h^k$ for some k.

Proof.

If both g, h are cyclically reduced then this is obvious.

Assume that g is not cyclically reduced and $g = xg_1x^{-1}$ with g_1 cyclically reduced. Then $g_1^k = h_1^k$ where h_1 is the reduced word $\sim x^{-1}hx$.

$$g_1 \text{ cyclically reduced } \Leftrightarrow g_1^k \text{ cyclically reduced}$$

Since $h_1^k = g_1^k$ we must also have that h_1 is cyclically reduced. So $h_1 = g_1$. Hence $h = g$. $\square$

Finitely generated, finitely presented groups

Isomorphism problem: Given $G = \langle X \rangle$ and $G' = \langle Y \rangle$, determine if $G \simeq G'$. For free groups, $F(X)$, $F(Y)$, this is settled.

We now define a general class of groups for which the three problems can be formulated, i.e. groups that are describable by finite data, i.e. finitely presented.

Suppose $G = \langle X \rangle$, $|X| < \infty$ (G finitely generated).

Remark

- *G finitely generated $\Rightarrow$ G countable.*
- *There exist uncountably many non-isomorphic f.g. groups.*

Algorithmic problems for infinite groups

Proposition

Suppose $G = \langle X \rangle$ with $|X| < \infty$, and suppose also that $G = \langle Y \rangle$. Then there exists some finite $Y_0 \subset Y$ such that $G = \langle Y_0 \rangle$.

Proposition

1. *If G finitely generated and $N \trianglelefteq G$, then G/N is finitely generated.*
2. *Finite generation is not inherited by subgroups (see Ex 2(iii) on Sheet 1: $F(\mathbb{N}) \leq F_2$).*
3. *Finite generation is inherited by finite index subgroups (Ex.).*
4. *Suppose we have a short exact sequence*

$$1 \to N \to G \to Q \to 1$$

and N, Q are finitely generated. Then G is finitely generated.

Presentations of groups

How to fully describe a group?

- Table of multiplication if G is finite;
- Free groups.

Answer in general case: by generators and relations.

Example

$\mathbb{Z}^2$ is the group generated by two elements a, b satisfying the relation

$$ab = ba \Leftrightarrow [a, b] = 1.$$

We write $\mathbb{Z}^2 = \langle a, b \mid [a, b] = 1\rangle$ or simply $\mathbb{Z}^2 = \langle a, b \mid [a, b]\rangle$.

Presentations of groups 2

In general, let $G = \langle S \rangle$. By Universal property, $\exists$ an onto homomorphism

$$\pi_S : F(S) \to G$$

whence G isomorphic to $F(S)/\ker(\pi_S)$.
The elements of $\ker(\pi_S)$ are called relators or relations for G and the generating set S.
We are interested in minimal subsets R of $\ker(\pi_S)$ such that $\ker(\pi_S)$ is normally generated by R.
$N \lhd G$ is normally generated by $R \subset N$ or N normal closure of R, $N = \langle\langle R \rangle\rangle$, if one of the following equivalent properties is satisfied:

- N is the smallest normal subgroup of G containing R;
- $N = \bigcap_{R \subset K \lhd G} K$;
- $N = \{r_1^{x_1} \cdots r_n^{x_n} \mid n \in \mathbb{N}, r_i \in R \cup R^{-1}, x_i \in G\} \cup \{1\}$.

Notation

$a^b = bab^{-1}$, $A^B = \{a^b \mid a \in A, b \in B\}$. *Then* $N = \langle\langle R \rangle\rangle \Leftrightarrow N = \langle R^G \rangle$

Presentation of groups 3

Let $R \subset \ker(\pi_S)$ be such that $\ker(\pi_S) = \langle\langle R \rangle\rangle$.
We say that the elements $r \in R$ are defining relators.
The pair (S, R) defines a presentation of G.
We write $G = \langle S \mid r = 1, \forall r \in R \rangle$ or simply $G = \langle S \mid R \rangle$.
Formally, it means G is isomorphic to $F(S)/\langle\langle R \rangle\rangle$.
Equivalently:

- $\forall g \in G$, $g = s_1 \cdots s_n$, for some $n \in \mathbb{N}$ and $s \in S \cup S^{-1}$;
- $w \in F(S)$ satisfies $w =_G 1$ if and only if in $F(S)$

$$w = \prod_{i=1}^{m} r_i^{x_i}, \text{ for some } m \in \mathbb{N}, r_i \in R, x_i \in F(S).$$

Examples of group presentations

1. $\langle a_1, \ldots, a_n \mid [a_i, a_j], 1 \leqslant i, j \leqslant n \rangle$ is a finite presentation of $\mathbb{Z}^n$;
2. $\langle x, y \mid y^2, yxyx \rangle$ is a presentation of the infinite dihedral group D_∞ ;
3. $\langle x_1, \ldots, x_{n-1} \mid x_i^2, [x_i, x_j] \text{ for } |j - i| \geq 2, (x_i x_{i+1})^3 \rangle$ is a presentation of the permutation group S_n.

Generalization of the Universal Property

Proposition

Let $G = \langle S|R\rangle$. Let H be a group and $\psi : S \to H$ be a map s.t. for every $r = s_1 \cdots s_n \in R$, $\psi(s_1)\cdots\psi(s_n) = 1$.
Then ψ has an unique extension to a group homomorphism $\Phi : G \to H$.

Proof: Exercise.

We are interested in groups with finite presentation.

Remark

Finitely presented groups compose a countable family of finitely generated groups.

It is important to understand if being finitely presented is an intrinsic feature of the group, or if it depends on a "good choice" of generating set.

(Finite) presentations of groups

Proposition

If $G = \langle S|R\rangle$ is finitely presented and $\langle X|Q\rangle$ is an arbitrary presentation with $|X|$ finite, then there exists some finite $Q_0 \subseteq Q$ such that $G = \langle X|Q_0\rangle$.

Proof: We have an isomorphism

$$\phi : F(S)/\langle\langle R\rangle\rangle \to F(X)/\langle\langle Q\rangle\rangle$$

Write $\phi(s) = \sigma_s$. Then $\forall x \in X$,

$$x = w_x(\{\sigma_s : s \in S\}) \quad \text{(with equality in } F(X)/\langle\langle Q\rangle\rangle)$$

So $x = w_x(\sigma_S)u_x$, $u_x \in \langle\langle Q\rangle\rangle$, with the equality being in $F(X)$. Let $r \in R$, and write $v_r = r(\{\sigma_s : s \in S\}) \in \langle\langle Q\rangle\rangle$.

Let $T_0 \subseteq \langle\langle Q\rangle\rangle$ be the finite set $\{u_x, v_r : x \in X, r \in R\}$.