String Theory 1

Lecture #3

Chapter 1 Classical relativistic string

today we continue
study relativistic classical string propagating in a fixed spacetime M

1.1 Classical relativistic point particle lecture 2

1.2 Classical relativistic string: action principle

1.3 Classical solutions

1.3.1 EOM & boundary conditions

⋮

1.2 Classical relativistic string

Continued from Lecture #2

The Polyakov action:

(induced metric on $\Sigma \subset M$]

$$\boxed{S_P[\gamma_{ab}, X^\mu] = -\frac{T}{2}\int_\Sigma \sqrt{-\gamma}\,\gamma^{ab}\,\overbrace{\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}}^{h_{ab}}\, d\tau\, d\sigma}$$

$\boxed{\xi^a = (\tau, \sigma)}$

$\gamma_{ab}(\xi)$ Lorentzian world-sheet metric (NEW field on Σ]

$\gamma = \det(\gamma_{ab})$

EOM wrt δX^μ:

$$\partial_a\left(\sqrt{-\gamma}\,\gamma^{ab}\,g_{\mu\nu}(X)\,\partial_b X^\nu\right)=0$$

EOM for the WS metric γ

GR1 :

$$T_{ab} \equiv -\frac{2}{T}\,\frac{1}{\sqrt{-\gamma}}\,\frac{\delta S_P}{\delta \gamma^{ab}}$$

stress tensor or energy momentum tensor

$$\Rightarrow \quad T_{ab} = \underbrace{\partial_a X^\mu\,\partial_b X^\nu\,g_{\mu\nu}}_{h_{ab}} - \frac{1}{2}\gamma_{ab}\left(\gamma^{bc}\underbrace{\partial_b X^\mu\,\partial_c X^\nu\,g_{\mu\nu}}_{h_{bc}}\right) = 0$$

$$\Rightarrow \quad h_{ab} = \frac{1}{2}\left(\gamma^{bc}\,h_{bc}\right)\gamma_{ab}$$

$\gamma_{ab}(X) \propto$ pullback metric h_{ab}

using this in S_p one gets back S_{NG}:

in fact, the proportionality factor drops out of S_p

$$\sqrt{-\gamma}\,\gamma_{ab} = \sqrt{-h}\,h_{ab}$$

We get then S_{NG} & same EOM

$\therefore$ S_p & S_{NG} are equivalent classically

Ⓐ Symmetries of the Polyakov action

WS perspective → global symmetries

▶ spacetime isometries
(Poincaré invariance when M = Minkowski)
and γ does not transform

WS perspective → gauge symmetries

▶ Worldsheet reparametrisations $\xi^a \longmapsto \tilde{\xi}^a(\xi)$ diffeos of Σ

$$\gamma_{ab}(\xi) \longmapsto \tilde{\gamma}_{ab}(\tilde{\xi}) = \gamma_{cd}(\xi) \frac{\partial \tilde{\xi}^c}{\partial \xi^a} \frac{\partial \tilde{\xi}^d}{\partial \xi^b}$$ symmetric 2 tensor on Σ

&

$$X^\mu(\xi) \longmapsto \tilde{X}^\mu(\tilde{\xi}) = X^\mu(\xi)$$ (WS scalars)

local diffeomorphism invariance

∇ = Levi Civita for γ_{ab}

$\Longrightarrow$ $\nabla_a T^{ab} = 0$ when EOM are satisfied ("on shell")

(conservation equation!) E Noether: for each symmetry of the action, there is a corresponding conserved current

So far: these where already in S_{NG}

special to 2dims

▶ **Weyl invariance** ie local scale symmetry acting on the 2dim metric on Σ

$$\gamma_{ab} \longmapsto e^{2\omega(\xi)}\gamma_{ab}, \quad X^\mu \text{ invariant}$$

$$[\sqrt{-\gamma} \mapsto e^{2\omega}\sqrt{-\gamma} \; ; \; \gamma^{ab} \mapsto e^{-2\omega}\gamma^{ab}]$$

Weyl invariance is also a gauge symmetry

[Weyl invariance very important in quantisation: anomalous unless D=26!]

Why is Weyl invariance special in 2dims (special to the string)

Consider instead a p-dimensional extended object with (p+1)dim WV

factor $\gamma^{ab}\sqrt{-\gamma} \xrightarrow[\text{Weyl transf.}]{} e^{-2\omega}\gamma^{ab} e^{(p+1)\omega}\sqrt{-\gamma}$

$e^{(p-1)\omega}$

$\boxed{\text{not invariant unless } p=1 \text{ ie a string}}$

- Lack of Weyl inv for higher dim p-branes makes it harder to understand non-perturbative physics in strings (D-branes ...) & M-theory

Tracelessness of T_{ab}

There is an important consequence of Weyl invariance

$$\boxed{T_{ab} \text{ is traceless}}$$

Recall

$$T_{ab} = \underbrace{\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}}_{h_{ab}} - \frac{1}{2}\gamma_{ab}\left(\gamma^{bc}\underbrace{\partial_b X^\mu \partial_c X^\nu g_{\mu\nu}}_{h_{bc}}\right) = 0$$

$$\mathrm{Tr}\, T = T_{ab}\gamma^{ab} = \gamma^{ab}h_{ab} - \frac{1}{2}\cdot 2\cdot\gamma^{bc}h_{bc} = 0$$ automatically

so $\mathrm{Tr}\, T$ is not a constraint

$T_{ab} = 0$ only two EOM

Why is T_{ab} traceless? consequence of Weyl inv

Recall $\delta S = \frac{\delta S}{\delta \gamma^{ab}} \delta \gamma^{ab} \propto \sqrt{-\gamma}\, T^{ab}\, \underset{\sim}{\delta \gamma_{ab}}$

Consider an infinitesimal Weyl transformation

$$\gamma_{ab} \longmapsto e^{2\omega(\xi)} \gamma_{ab} = \left(1 + 2\omega(\xi) + \dots\right) \gamma_{ab}$$

so $\delta \gamma_{ab} = 2\omega(\xi)\gamma_{ab}$ $(-\delta \det \gamma = \det \gamma\, \gamma_{ab}\, \delta \gamma^{ab})$

$$\Rightarrow \underset{\text{wrt Weyl transf}}{\longrightarrow} \delta S \propto 2\sqrt{-\gamma}\, \omega(\xi)\, T^{ab} \gamma_{ab} \overset{\text{Weyl inv.}}{=} 0 \quad \text{for } \underline{\text{any}}\ \omega$$

$$\therefore \boxed{T_{ab}\gamma^{ab} = T^{a}{}_{a} = 0}$$

regardless of EOM

* (this does not require EOM!) *

S_p is 2 dim field theory describing D, 2dim scalar fields $X^\mu(\xi)$ coupled to the WS metric γ_{ab} ie 2dim gravity coupled to scalars

This begs the question {   How general is S_p?

Can one add terms to the action which are

- compatible with power counting renormalizability (at most 2 derivs)

and

- consistent with the symmetries of the action

Two possible terms (for the closed string and no other fields)

* $$S_{HE} = \frac{\lambda_2}{4\pi} \int_\Sigma \sqrt{-\gamma}\, \underbrace{R^{(2)}(\gamma)}_{\text{WS Ricci scalar}}\, d\tau\, d\sigma$$ ⇝ Hilbert-Einstein term for 2dim gravity

Integrand is (locally) a total derivative $\boxed{\text{PS 1}}$

⇒ does not affect the classical equations of motion

S_{HE} is topological (it depends only on the global topology of Σ)

in fact $S_{HE} = \lambda_2 \chi(\Sigma)$ (related to a coupling constant!)

(open string: Σ has boundaries and there is an extra term)

$\boxed{\text{Ignore for now}}$ but it is an important term in string perturbation theory!
when topologies of WS are important

* $$S_{CT} = \lambda \int_\Sigma \underbrace{\sqrt{-\gamma}\, d\tau\, d\sigma}$$ ⇝ cosmological constant term on Σ

area element inv under Reparams but not Weyl invariant

⇒ inconsistent EOM (BBS exercise) ⇒ $\lambda = 0$

(any term $\sim \int \sqrt{-\gamma}\, V(X)\, d\tau\, d\sigma$ not Weyl invariant)

(B) Gauge fixing the Polyakov action

As is usual in theories with gauge symmetries one can "chose" a gauge to simplify the action.

(choose a convenient gauge to simplify the action)

reparametrizations: $\gamma_{ab} \to e^{2\omega(\tau,\sigma)} \eta_{ab}$ conformal gauge

3 independent degrees freedom

$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

use a reparametrization (which involves 2 functions of coords) to gauge away 2 components of γ

Weyl: $e^{2\omega(\tau,\sigma)} \eta_{ab} \to \eta_{ab}$ unit gauge

(use a Weyl transformation to gauge away the remaining degree of freedom

$\boxed{\text{Remark}}$: locally one can prove that one can always choose this gauge $\gamma_{ab} = \eta_{ab}$.

However we do not know if this can be done globally on Σ !

There are in fact topological obstructions which are better understood in Euclidean signature.

To deal with Lorentian signatures one does a "Wick rotation" to Euclidean signature

See BLT p18 for a discussion

Polyakov action in conformal gauge ; M = Minkowski

indeed it simplifies drastically

$$\boxed{S_P^{CG}[X^\mu] = -\frac{T}{2}\int\left(-\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X\right) d\tau d\sigma = -\frac{T}{2}\int \partial_a X\cdot\partial_b X\,\eta^{ab}\, d\tau d\sigma}$$

$\Rightarrow$ theory of D massless scalar fields in flat (1+1)-dim space (though one term with the wrong sign)

EOM (for X^μ) :

simplifies !

$$\boxed{\partial_a\left(g_{\mu\nu}\partial^a X^\nu\right) = 0}$$

$\partial_a\partial^a X^\mu = 0$

when $g_{\mu\nu} = \eta_{\mu\nu}$

(wave equation !)

EOM for γ: recall $T_{ab} = \partial_a X \cdot \partial_b X - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X \cdot \partial_d X$

$T_{ab} = 0$ become constraints after gauge fixing

In the conformal gauge

$$T_{ab} = \partial_a X \cdot \partial_b X - \frac{1}{2}\eta_{ab}\,\partial_c X \cdot \partial^c X = 0$$

$$T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2}\left(\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X\right) = 0$$

$$T_{\tau\sigma} = \partial_\tau X \cdot \partial_\sigma X = 0$$

tracelessness of T_{ab}: $T^a{}_a = \eta^{ab}T_{ab} = -T_{\tau\tau} + T_{\sigma\sigma} \overset{\text{automatically}}{=} 0$

$\hookrightarrow$ should hold irrespective of these constraints

2 constraints (instead of 3)

In summary:

$$S_P^{CG}[X] = -\frac{T}{2}\int_\Sigma \partial_a X\cdot\partial_b X\,\eta^{ab}\,d\tau\,d\sigma$$

gauge fixed Polyakov action

EOM

$$\partial_a\left(g_{\mu\nu}\partial^a X^\nu\right) = 0$$

$$T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2}\left(\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X\right) = 0$$

$$T_{\tau\sigma} = \partial_\tau X\cdot\partial_\sigma X = 0$$

Conservation of T_{ab}: $\nabla_a T^{ab} = 0$

1.3 Classical Solutions of S_P

$\boxed{g_{\mu\nu} = \eta_{\mu\nu}}$

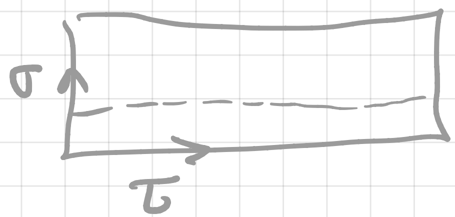
We are interested in solving the equations of motion for the fields X^μ which in the conformal gauge: $\partial_a \partial^a X^\mu = 0$ 2 dim wave eq

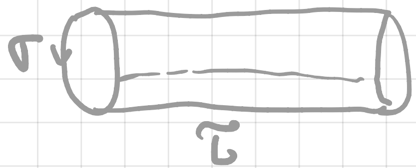
together with constraints coming from T_{ab}

For a single string propagating without sources we describe the string by considering

$\tau \to$ time coordinate on Σ

$-\infty \leq \tau \leq \infty$





$\sigma \to$ spatial coordinate on Σ

strings with finite spatial length $\sigma \in [0, \ell]$

1.3.1 Equations of motion and boundary conditions

Writing the action as $S[X] = \int_\Sigma d\tau\, d\sigma\, \mathcal{L}[X^\mu, \partial_a X^\mu]$

a standard computation gives

in classical field theory

$$\delta S = \int_\Sigma d\tau\, d\sigma \left\{ \frac{\partial \mathcal{L}}{\partial X^\mu} \delta X^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \delta \partial_a X^\mu \right\}$$

Π^a_μ : conjugate momentum

$$= \int_\Sigma d\tau\, d\sigma \left\{ \underbrace{\partial_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \delta X^\mu \right)}_{\text{total derivative}} + \underbrace{\left[\frac{\partial \mathcal{L}}{\partial X^\mu} - \partial_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \right) \right]}_{\hookrightarrow \text{ Euler-Lagrange (EOM)}} \delta X^\mu \right\}$$

$\delta S = 0$: 1st term must vanish too!

For the Polyakov action: $S_P^{CG}[X^\mu] = -\frac{T}{2}\int_\Sigma d\tau\, d\sigma\, \partial_a X \cdot \partial^a X$

- Euler-Lagrange equations (EOM)

$$0 = \partial_a \left(-\frac{T}{2} \cdot 2\eta^{ab}\partial_b X^\mu \right)$$

(note $\frac{\partial \mathcal{L}}{\partial X^\mu} = 0$)

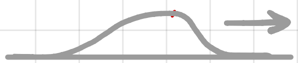
$$\eta^{ab}\partial_a\partial_b X^\mu = -\partial_\tau^2 X^\mu + \partial_\sigma^2 X^\mu = 0$$

two dim wave eq for waves travelling at $c=1$

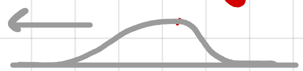
General solution: $X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$

(prelims)

R-moving



L-moving



wavefronts

use light-cone coords: $\xi^\pm = \tau \pm \sigma$

$$\partial_\pm = \frac{\partial}{\partial \xi^\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \;;\quad d\tau\, d\sigma = d\xi^+ d\xi^- \frac{\partial(\tau,\sigma)}{\partial(\xi^+,\xi^-)} = \frac{1}{2} d\xi^+ d\xi^-$$

$$\gamma_{++} = \gamma_{--} = \gamma^{++} = \gamma^{--} = 0 \;;\quad \gamma_{+-} = \gamma_{-+} = -\frac{1}{2} \;;\quad \gamma^{+-} = \gamma^{-+} = -2$$

$\gamma_{ab} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\gamma^{ab} = -2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

total derivative term

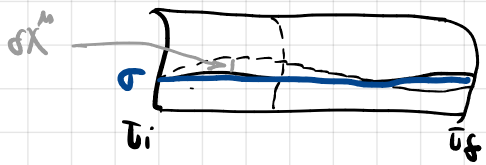
$$\int_\Sigma d\tau\, d\sigma\ \partial_a \left(\boxed{\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)}}\ \delta X^\mu \right) = 0$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)}} = -T\,(\partial_a X^\nu)\,\eta_{\mu\nu}$$

$$0 = -T \int_{\tau_i}^{\tau_f} d\tau \int_0^\ell d\sigma \left\{ \frac{\partial}{\partial \tau}\left(\eta_{\mu\nu}\, \partial_\tau X^\mu\, \delta X^\nu\right) + \frac{\partial}{\partial \sigma}\left(\eta_{\mu\nu}\, \partial_\sigma X^\mu \delta X^\nu\right) \right\}$$

first term (int wrt τ) : $-T \int_0^\ell d\sigma\, \eta_{\mu\nu} \left(\partial_\tau X^\mu\, \delta X^\nu\right) \Big|_{\tau=\tau_i}^{\tau=\tau_f} = 0$ because

$$\delta X^\mu(\tau_i, \sigma) = 0 \qquad \delta X^\mu(\tau_f, \sigma) = 0$$



ie string is kept fixed at initial & final positions

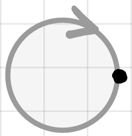
ie variations of the WS with fixed initial (at $\tau = \tau_i, \tau_f$) conditions

(analogy: particle $\delta X(\tau_i)=0$  $\delta X(\tau_f)=0$
sum of trajectory with fixed initial & final positions)

So the second term must vanish too:

$$0 = -T \int_{\tau_i}^{\tau_f} d\tau \, \left(\eta_{\mu\nu} \, \partial_\sigma X^\mu \, \delta X^\nu\right)\Big|_{\sigma=0}^{\sigma=\ell}$$

closed strings: this vanishes due to the

periodicity conditions  $\sigma \sim \sigma + \ell$

same point in M

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \ell)$$

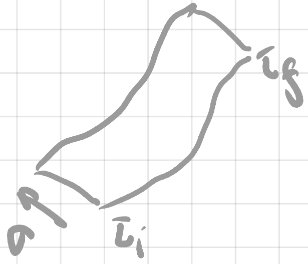
Moreover, solutions of the EOM are required to be periodic in σ with period ℓ

Open strings

boundary conditions on the string endpoints.

$$0 = -T \int_{\tau_i}^{\tau_f} d\tau \, \left(\eta_{\mu\nu} \, \partial_\sigma X^\mu \delta X^\nu \right) \Big|_{\sigma=0}^{\sigma=\ell} \rightsquigarrow \quad \partial_\sigma X_\mu \, \delta X^\mu = 0$$

at $\sigma = 0, \ell$



There are two natural choices:

Neumann: End points move freely in $\mathcal{M}$

(no constraints on δX^μ at $\sigma = 0, \ell$)

Then: $\partial_\sigma X^\mu (\tau, \ell) = 0$ & $\partial_\sigma X^\mu (\tau, 0) = 0$

Dirichlet $\qquad \delta X^\mu = 0 \quad$ at $\quad \sigma = 0, \ell$

ie ends of string **fixed** in $\mathcal{M}$

ie $\qquad X^\mu(\tau, \ell) = x_0^\mu(\tau), \qquad X^\mu(\tau, 0) = x_\ell^\mu(\tau)$

This involves a **choice** of spacetime vectors

$\Longrightarrow$ break Poincaré invariance

To be continued...

Next:

1.3 Classical solutions (continue)

1.3.1 boundary conditions (continue)

1.3.2 solutions of EOM + boundary conditions

1.3.3 Imposing the constraints $T_{ab}=0$ & $\partial_a T^{ab}=0$

1.3.4 The Witt-algebra and conformal symmetries

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