String Theory 1

Lecture #4

Chapter 1 Classical relativistic string

↳ study relativistic classical string propagating in a fixed spacetime M

1.1 Classical relativistic point particle

1.2 Classical relativistic string: action principle

1.3 Classical solutions

1.3.1 EOM & boundary conditions

1.3.2 Conserved charges associated to the symmetries of the action

1.3.3 Solutions of EOM + bound. conds

1.3.4 Satisfying the constraints

1.3.5 The Witt-algebra & conformal symmetries

1.3 Classical solutions continued

$\gamma_{ab} = \eta_{ab}$

$$S_P^{CG}[X] = -\frac{T}{2}\int_\Sigma \partial_a X \cdot \partial_b X \, \eta^{ab} \, d\tau \, d\sigma$$

gauge fixed Polyakov action

target space: $M = \mathbb{M}_D$ D-dim Minkowski

EOM $\quad \partial_a(\partial^a X^\nu) = 0 \; : \quad X^\mu(\xi) = X^\mu_L(\xi^+) + X^\mu_R(\xi^-) \qquad \xi^\pm = \tau \pm \sigma$

- impose conditions coming from the boundary term in $\delta S = 0$

- constraints from EOM for γ

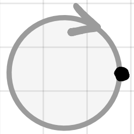
$$\begin{cases} T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2}(\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X) = 0 \\ T_{\tau\sigma} = \partial_\tau X \cdot \partial_\sigma X = 0 \end{cases}$$

- Conservation of T_{ab} : $\qquad \partial_a T^{ab} = 0$

1.3.1 Boundary conditions (continued)

- $\delta X^\mu(\tau_i,\sigma)=0$ & $\delta X^\mu(\tau_f,\sigma)=0$ string kept fixed at initial & final position
- $0 = \int_{\tau_i}^{\tau_f} d\tau \; \partial_\sigma X \cdot \delta X \Big|_{\sigma=0}^{\sigma=\ell}$

closed strings periodicity conditions



$\sigma \sim \sigma+\ell$

ie $$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\ell)$$

so boundary term vanishes

Moreover: solutions of the EOM must be periodic in σ with period ℓ

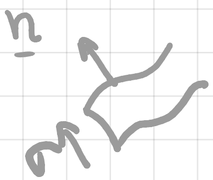
open strings: boundary conditions on the string endpoints

$$0 = -T \int_{\tau_i}^{\tau_f} d\tau \, \left(\partial_\sigma X \cdot \delta X \right) \Big|_{\sigma=0}^{\sigma=\ell} \rightsquigarrow \boxed{\partial_\sigma X_\mu \, \delta X^\mu = 0 \quad \text{at} \quad \sigma = 0, \ell}$$

- **Neumann (NN)** no constraints on δX^μ at $\sigma = 0, \ell$ so endpoints move freely in $\mathcal{M}$

as long as: $\partial_\sigma X^\mu(\tau, \ell) = 0$ & $\partial_\sigma X^\mu(\tau, 0) = 0$

"no momentum flowing off the string"



momentum $\perp$ to $\partial\Sigma$

- **Dirichlet** (DD) $\delta X^\mu = 0$ at $\sigma = 0, \ell$ — ends of string fixed in M

  ie $X^\mu(\tau, \ell) = x^\mu_\ell(\tau)$, $X^\mu(\tau, 0) = x^\mu_0(\tau)$

This involves a choice of spacetime vectors $\Rightarrow$ break Poincaré invariance

- One can have mixed boundary conditions: for example

  - Neumann (NN) on $p+1$ words

    Dirichlet (DD) on $D-(p+1)$ words

The ends of the string are free to move only on a subspace $\mathcal{Q} \subset M$, $\dim \mathcal{Q} = p+1$.
This subspace is called a Dp-brane with x^μ_0 & x^μ_ℓ interpreted as the position of the brane.

(very important! needed for internal consistency of the non-perturbative theory; more later)

- One can also have (ND) boundary conditions.

1.3.2 Solutions of EOM + bound. conds.

general solution of the wave eq $X^\mu(\tau,\sigma) = X_R^\mu(\xi^-) + X_L^\mu(\xi^+)$

Closed strings: $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\ell)$, $(\xi^\pm \to \xi^\pm \pm \ell)$

Expand in Fourier modes:

separately periodic up to a two mode

$$X_L^\mu(\xi^+) = \frac{1}{2}x^\mu + \frac{\pi}{\ell}\alpha' p^\mu \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-\frac{2\pi i}{\ell} n \xi_+}$$

$$X_R^\mu(\xi^-) = \frac{1}{2}x^\mu + \frac{\pi}{\ell}\alpha' p^\mu \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\alpha_n^\mu e^{-\frac{2\pi i}{\ell} n \xi^-}$$

where x^μ, p^μ, $\tilde{\alpha}_n^\mu$ and α_n^μ are the Fourier coeffs.

X^μ is real-valued: $x^\mu \in \mathbb{R}$, $p^\mu \in \mathbb{R}$, $\tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*$, $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$

$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$ from periodicity $\sigma \to \sigma + \ell$

B Zwiebach has very detailed solution (Chapter 7)

- $X^\mu(\tau, \sigma+\ell) = X_L^\mu(\xi^+ + \ell) + X_R^\mu(\xi^- + \ell)$ $\qquad \xi^\pm \mapsto \xi^\pm \pm \ell$

$$X^\mu(\tau,\sigma) = X_L^\mu(\xi^+) + X_R^\mu(\xi^-)$$

$\Rightarrow \quad X_L^\mu(\xi^+ + \ell) - X_L^\mu(\xi^+) = -\left(X_R^\mu(\xi^- - \ell) - X_R^\mu(\xi^-)\right) = \text{constant}$

$$= \frac{\pi}{\ell}\alpha' p^\mu \ell .$$

$\Rightarrow \quad X_{L,R}^\mu(\xi^\pm) = \frac{1}{2}x^\mu + \frac{\pi}{\ell}\alpha' p^\mu \xi^\pm + \quad \cdots$

Useful for later:

$$\partial_+ X^\mu(\xi^+) = \partial_+ X_L^\mu = \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^\mu \, e^{-\frac{2\pi i}{\ell} n \xi_+}$$

$$\partial_- X^\mu(\xi^-) = \partial_- X_R^\mu = \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu \, e^{-\frac{2\pi i}{\ell} n \xi_-}$$

where $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$

Prefactors for convenient physical interpretations as we will see below

Open strings with Neumann (NN) boundary conditions

$$\partial_\sigma X^\mu(\tau, \ell) = 0 \qquad \partial_\sigma X^\mu(\tau, 0) = 0$$

Due to the boundary conditions X_L^μ & X_R^μ are no longer independent $(\tilde{\alpha}_n^\mu = \alpha_n^\mu)$

$$X^\mu(\tau,\sigma) = \underbrace{x^\mu + \frac{2\pi\alpha'}{\ell} p^\mu \tau}_{\text{average position}} + i\sqrt{2\alpha'} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n} \alpha_n^\mu e^{-\frac{i\pi n}{\ell}\tau} \cos\left(\frac{n\pi\sigma}{\ell}\right)$$

$$x^\mu(\tau) = \frac{1}{\ell}\int_0^\ell d\sigma X^\mu(\tau,\sigma)$$

X^μ real-valued : $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$

$$\partial_\pm X^\mu = \frac{\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-\frac{i\pi n}{\ell}\xi^\pm}, \qquad \alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$$

See lecture notes for open strings with DD & ND boundary conds

1.3.3 Conserved charges

Recall Noether's theorem: for each symmetry in the action there is a corresponding conserved current. We also have Noether charges

▶ (global) symmetries corresponding to the isometries of M: Poincaré invariance

$$X^\mu \longmapsto \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz transfs}} X^\nu + V^\mu$$

translations

conserved current $\quad \Pi^\mu_a = \dfrac{\delta \mathcal{L}}{\partial \partial_a X^\mu}$

momentum conjugate

$(\partial_a \Pi^a_\mu)\eta^{ab} = 0$

- translations $X^\mu(\xi) \mapsto X^\mu(\xi) + V^\mu$

current $q^a_\mu = -T\sqrt{-\gamma}\,\gamma^{ab}\partial_a X_\mu = -T\eta^{ab}\partial_b X_\mu$

conservation $\partial_a q^a{}_\mu = 0$

conservation of the energy momentum current

charges: $\int_0^\ell (q^\mu)^\tau d\sigma = T\int_0^\ell \partial_\tau X^\mu \equiv p^\mu$ center of mass momentum

(spatial integral of the τ-component of each current)

- Lorentz transformations: $X^\mu \longrightarrow \Lambda^\mu{}_\nu X^\nu$

current $J^a{}_{\mu\nu} = -T\eta^{ab}(X_\mu \partial_b X_\nu - X_\nu \partial_b X_\mu) = X_\mu q^a_\nu - X_\nu q^a_\mu$

conservation $\partial_a J^a_{\mu\nu} = 0$

conservation of angular momentum current

charges: $\frac{1}{\ell}\int_0^\ell (J^{\mu\nu})^\tau d\sigma$

- WS-symmetries: WS diffeomorphisms (later)

conserved current T_{ab}, $\eta^{ab} \partial_a T_{bc} = 0$

Interpretation of the coefficients

translations

$$\int_0^\ell d\sigma\, (q^\mu)^\tau = T \int_0^\ell \partial_\tau X^\mu = \begin{cases} p^\mu & \text{closed string \& open string with (NN) bound. conds} \\ 0 & \text{open string with (DD) bound. cond} \end{cases}$$

$\partial_\tau X^\mu = \frac{2\pi\alpha'}{\ell} p^\mu +$ terms that vanish upon τ-integration

p^μ : centre of mass momentum of the string

Lorentz transformations

$$M^{\mu\nu} = \int_0^{\ell} d\sigma \, (J^{\mu\nu})^{\tau} = \begin{cases} \ell^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} & \text{closed strings} \\ \ell^{\mu\nu} + E^{\mu\nu} & \text{(NN) open string} \end{cases}$$

$J^{\mu\nu}_{a} = -T(X^{\mu}\partial_a X^{\nu} - X^{\nu}\partial_a X^{\mu})$

$$\ell^{\mu\nu} = x^{\mu} p^{\nu} - x^{\nu} p^{\mu}$$

centre of mass contribution

two mode of spacetime angular momentum

$$E^{\mu\nu} = -i \sum_{n \neq 0} \frac{1}{n} \left(\alpha^{\mu}_{-n} \alpha^{\nu}_{n} - \alpha^{\nu}_{-n} \alpha^{\mu}_{n} \right)$$

$$\tilde{E}^{\mu\nu} = -i \sum_{n \neq 0} \frac{1}{n} \left(\tilde{\alpha}^{\mu}_{-n} \tilde{\alpha}^{\nu}_{n} - \tilde{\alpha}^{\nu}_{-n} \tilde{\alpha}^{\mu}_{n} \right)$$

contribution of L & R moving waves to the spacetime angular momentum

1.3.4 Satisfying the constraints

Recall that we need to impose constraints from the stress tensor. In the light-cone coordinates

- conservation: $\eta^{ab}\partial_a T_{bc} = 0 \implies \partial_+ T_{--} + \partial_- T_{+-} = 0$

$$\partial_- T_{++} + \partial_+ T_{-+} = 0$$

tracelessness $\quad \eta^{ab} T_{ab} = 0 \implies T_{+-} + T_{-+} = 0$

$$\implies T_{+-} = T_{-+} \quad \text{symmetric tensor}$$

$$\underline{T_{+-} = 0}$$

automatic

$$\implies \underline{\partial_+ T_{--} = 0 \qquad \partial_- T_{++} = 0}$$

These are extremely powerful!

* 2dim Lorentzian version of holomorphicity (antiholomorphicity). These give us our $\boxed{\text{infinite}}$ set of conserved charges! *

- Finally enforce $\underline{T_{++} = 0 \qquad T_{--} = 0}$

Ⓐ Closed strings

Let $f(\xi^-)$ be an arbitrary function and consider

$$Q_f = \int_0^\ell d\sigma \, f(\xi^-) \, T_{--}(\xi^-) \qquad \partial_+ T_{--} = 0$$

$$\partial_\pm = \tfrac{1}{2}(\partial_\tau \pm \partial_\sigma)$$
so $\partial_\tau = 2\partial_+ - \partial_\sigma$

$$\Rightarrow \frac{\partial}{\partial\tau} Q_f = \int_0^\ell d\sigma \, (\cancel{2\partial_+} - \partial_\sigma)(f(\xi^-) T_{--}(\xi^-))$$

$$= - \int_0^\ell d\sigma \, \partial_\sigma (f(\xi^-) T_{--}(\xi^-)) = - (f(\xi^-) T_{--}(\xi^-)) \Big|_{\sigma=0,\, \tau \text{ fixed}}^{\sigma=\ell,\, \tau \text{ fixed}}$$

$= 0$ if $f(\xi^-)$ is periodic

That is: the current $f(\xi^-) T_{--}(\xi^-)$ is also conserved!

Since f is arbitrary $\Rightarrow$ there is an infinite set of conserved currents

Similarly: T_{++} is conserved and so is $g\, T_{++}$, $g = g(\xi^+)$ periodic

A complete set of periodic functions in σ is given by

$$\left\{ f_m(\xi^-) = e^{\frac{2\pi i}{\ell} m \xi^-} \;, \quad m \in \mathbb{Z} . \right\}$$

Define then an infinite set of charges:

$$L_m = \frac{T\ell}{2\pi} \int_0^\ell d\sigma \, e^{\frac{2\pi i}{\ell} m \xi^-} \, T_{--}(\xi^-) = \frac{T}{2\ell} \int_0^\ell d\sigma \, e^{\frac{2\pi i}{\ell} m \xi^-} \partial_- X_R \partial_- X_R$$

$(T_{--} = \partial_- X \cdot \partial_- X = \partial_- X_R \cdot \partial_- X_R)$

Fourier modes of T_{--}

Using the mode expansion for X^μ $\left(\partial_- X_R^\mu = \frac{2\pi}{\ell} \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-\frac{2\pi i}{\ell} n \xi^-} \right)$

take $\tau = 0$ WLOG as L_m are conserved

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \qquad \text{with} \qquad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} \, p^\mu$$

Note that $L_{-m} = (L_m)^*$ because T_{--} is real similarly: for $T_{++}(\xi^+)$, $g_m(\xi^+) = e^{\frac{2\pi i}{\ell} m \xi^+}$

$$\tilde{L}_m = \frac{T\ell}{2\pi} \int_0^\ell d\sigma \; e^{\frac{2\pi i}{\ell} m \xi^+} \; T_{++}(\xi^+)$$

Fourier modes of T_{++}

$$= \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \, ,$$

$\tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} \, p^\mu$

We have then

$$\boxed{L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n \qquad \tilde{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n}$$

Recall that L_m & $\tilde{L}_m$ are the Fourier components of T_{--} & T_{++} respectively. Then setting

$$\boxed{L_m = 0 \quad , \quad \tilde{L}_m = 0} \qquad \forall\ m\in\mathbb{Z}$$

imposes the constraints $T_{++}=0$ & $T_{--}=0$

* The vanishing of these charges is equivalent to quadratic constraints on the oscillators *

$$\{x^\mu, p^\mu, \alpha_n^\mu, \tilde{\alpha}_n^\mu\}$$

Consider these constraints for L_0 & $\tilde{L}_0$ (particularly interesting)

$$L_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot \alpha_n = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \sum_{n=1} \alpha_{-n} \cdot \alpha_n \qquad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$$

$$= \frac{\alpha'}{4} p^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = 0 \qquad (p^2 = p \cdot p = p^\mu p_\mu)$$

$$\tilde{L}_0 = \frac{\alpha'}{4} p^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = 0$$

Recall p^μ = spacetime centre of mass momentum

- We have a mass shell condition : $M^2 = -p^2$

$$-p^2 = M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n)$$

contributions of osc. modes to the effective mass of the string in spacetime

rest mass^2 of a string in a given state of oscillation

- Moreover : $\displaystyle\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = -\frac{\alpha'}{4} p^2$

level matching condition
(relates L/R modes)

(do not forget we still need to enforce $L_m = 0$, $\tilde{L}_m = 0$ $\forall m \neq 0$)

(B) **Open strings** with (NN) boundary conditions on all X^μ

Let $f(\xi^+)$ & $g(\xi^-)$ be arbitrary functions. Then

$$\partial_+\left(g(\xi^-)\,T_{--}(\xi^-)\right)=0 \quad \& \quad \partial_-\left(f(\xi^+)\,T_{++}(\xi^+)\right)=0$$

Let

$$Q_{f,g}=\int_0^\ell d\sigma\,\left(f(\xi^+)\,T_{++}+g(\xi^-)\,T_{--}\right)$$

and seek conditions on f & g st Q is conserved:

$$\partial_\tau Q_{f,g}=\int_0^\ell d\sigma\,\left(\partial_\sigma\left(f(\xi^+)\,T_{++}\right)-\partial_\sigma\left(g(\xi^-)\,T_{--}\right)\right)$$

$\partial_\tau=2\partial_-+\partial_\sigma$ $\qquad$ $\partial_\tau=2\partial_+-\partial_\sigma$

$$=\left(f(\xi^+)\,T_{++}-g(\xi^-)\,T_{--}\right)\Big|_0^\ell$$

At $\sigma=0,\ell$: $\quad \partial_+X^\mu=\partial_-X^\nu$ so $T_{++}=T_{--}$

$\partial_\pm X^\mu=\frac{\pi}{\ell}\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu\, e^{-\frac{i\pi n}{\ell}\xi^\pm}$

$\Rightarrow$ Q is conserved if $f(\xi^+) = g(\xi^-)$ at $\sigma = 0, \ell$

(i) $\sigma = 0 \Rightarrow f(\tau) = g(\tau)$ (f & g are the same function)

(ii) $\sigma = \ell \Rightarrow f(\tau+\ell) = g(\tau-\ell) = f(\tau-\ell)$

$\therefore f(\tau) = f(\tau+2\ell)$

f periodic function with period 2ℓ

Let $f(\xi^+) = e^{\pi i m \xi^+/\ell}$ $\quad g(\xi^-) = e^{\pi i m \xi^-/\ell}$

and define

$$L_m = \frac{T\pi}{\ell}\int_0^\ell d\sigma \left(e^{\pi i m \xi^+/\ell} T_{++} + e^{\pi i m \xi^-/\ell} T_{--}\right) \overset{\tau=0}{=} \frac{T\pi}{\ell}\int_0^\ell d\sigma \left(e^{\pi i m \sigma/\ell} T_{++}(\sigma) + e^{-\pi i m \sigma/\ell} T_{--}(\sigma)\right)$$

$$\Rightarrow \quad \boxed{L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n} \quad \text{with} \quad \alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$$

L_0 gives open-string mass-shell condition

$L_0 = 0$: $\quad M^2 = -p^2 = \frac{1}{\alpha'} \sum_{n=1} \alpha_{-n} \cdot \alpha_n$

rest mass of a string due to the oscillators

Summary

We have constructed explicitly the space of solutions of the eqs of motion ie, the phase space.

This is an infinite dimensional affine space with coordinates

$$\{ x^\mu, p^\mu, \alpha_n^\mu, \tilde{\alpha}_n^\mu \}$$

subject to quadratic constraints

$$\{ L_n = 0, \tilde{L}_n = 0, \forall n \in \mathbb{Z} \}$$

for the closed string; for the open string neglect $\tilde{\alpha}$, $\tilde{L}$

where L_n (& $\tilde{L}_n$) constitute an infinite set of conserved charges corresponding to the Fourier modes of T_{ab}

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \quad , \quad \tilde{L}_m = \cdots$$

Next

↳

1.3.5 The Witt-algebra & conformal symmetries

L_n (& $\bar{L}_n$) satisfy an algebra

algebra of generators of conformal transformations