Geometric Group Theory

Cornelia Druțu

University of Oxford

Part C course HT 2025

Reference for the small cancellation technique

Alexander Yu. Olshanskii, *Geometry of defining relations in groups*. Mathematics and its Applications 70. Kluwer Academic Publishers Group, Dordrecht, 1991.

(Finite) presentations of groups

Proposition

If $G = \langle S | R \rangle$ is finitely presented and $\langle X | Q \rangle$ is another presentation with $|X|$ finite, then there exists $Q_0 \subseteq Q$ finite such that $G = \langle X | Q_0 \rangle$.

Proof: We have an isomorphism

$$\phi : F(S)/\langle\langle R \rangle\rangle \to F(X)/\langle\langle Q \rangle\rangle$$

Write $\phi(s) = \sigma_s(X)$. Likewise, $\forall x \in X$, $\phi^{-1}(x) = w_x(S)$, hence

$$x = w_x(\{\sigma_s : s \in S\}) \quad \text{(with equality in } F(X)/\langle\langle Q \rangle\rangle).$$

So $x = w_x(\sigma_S)u_x$ in $F(X)$, for some $u_x \in \langle\langle Q \rangle\rangle$.
$\forall r \in R$, write $v_r = r(\{\sigma_s : s \in S\}) \in \langle\langle Q \rangle\rangle$.

Let $T_0 \subseteq \langle\langle Q \rangle\rangle$ be the finite set $\{u_x, v_r : x \in X, r \in R\}$.

(Finite) presentations of groups

Let $T_0 \subseteq \langle\langle Q \rangle\rangle$ be the finite set $\{u_x, v_r : x \in X, r \in R\}$.

Claim: $\langle\langle T_0 \rangle\rangle = \langle\langle Q \rangle\rangle$.

Proof of claim: Define

$$f : F(S)/\langle\langle R \rangle\rangle \to F(X)/\langle\langle T_0 \rangle\rangle, \quad f(s) = \sigma_s.$$

Then f is an onto homomorphism.

Also, given $\pi : F(X)/\langle\langle T_0 \rangle\rangle \to F(X)/\langle\langle Q \rangle\rangle$, $\pi \circ f = \phi$ is an isomorphism and hence f is injective.

This proves the claim. Whence $G = \langle X \mid T_0 \rangle$. But T_0 is not a subset of Q. Every $\rho \in T_0 \subseteq \langle\langle Q \rangle\rangle$ can be written as $\rho = \prod_{r \in F_\rho} r^{x_r}$ in $F(X)$, where $F_\rho \subset Q$ finite. Take $Q_0 = \bigcup_{\rho \in T_0} F_\rho$ finite subset of Q.

Then $\langle\langle T_0 \rangle\rangle \subseteq \langle\langle Q_0 \rangle\rangle \subseteq \langle\langle Q \rangle\rangle$, whence $\langle\langle Q_0 \rangle\rangle = \langle\langle Q \rangle\rangle$. It follows that $G = \langle X \mid Q_0 \rangle$. □

Tietze transformations

How do we recognise when two finite presentations give the same group?

There are two types of transformations (called Tietze transformations).

(T1) Given $\langle S|R\rangle$ and $r \in \langle\langle R\rangle\rangle$, change the presentation to $\langle S|R \cup \{r\}\rangle$ (or do the inverse operation).

(T2) Given $\langle S|R\rangle$, a new symbol $a \notin S$ and $w \in F(S)$, change the presentation to $\langle S \cup \{a\}|R \cup \{a^{-1}w\}\rangle$ (or do the inverse operation).

Theorem

Two finite presentations define isomorphic groups if and only if they are related by a finite sequence of Tietze transformations.

Proof: $(\Leftarrow)$ (T1) defines isomorphic groups because $\langle\langle R\rangle\rangle = \langle\langle R \cup \{r\}\rangle\rangle$.

Tietze transformations

Theorem

Two finite presentations define isomorphic groups if and only if they are related by a finite sequence of Tietze transformations.

Proof continued: For (T2), consider the homomorphisms

$$\iota : F(S) \hookrightarrow F(S \cup \{a\}) \quad \text{(injection)}$$
$$f : F(S \cup \{a\}) \twoheadrightarrow F(S) \quad f(a) = w \quad \text{(surjection)}$$

Note that $f \circ \iota = \mathrm{id}_{F(S)}$. They induce homomorphisms

$$F(S) \xrightarrow{\bar{\iota}} F(S \cup \{a\}) / \langle\langle a^{-1}w \rangle\rangle \xrightarrow{\bar{f}} F(S)$$

with $\bar{f} \circ \bar{\iota} = \mathrm{id}_{F(S)}$. $\bar{\iota}$ is onto, and hence $\bar{\iota}$ and $\bar{f}$ are isomorphisms. Since also $\bar{f}^{-1}(\langle\langle R \rangle\rangle) = \langle\langle R \cup \{a^{-1}w\} \rangle\rangle / \langle\langle a^{-1}w \rangle\rangle$ we have that $\bar{f}$ induces the desired isomorphism.

Tietze transformations

Theorem

Two finite presentations define isomorphic groups if and only if they are related by a finite sequence of Tietze transformations.

Proof continued:

($\Rightarrow$) Let $G_1 = \langle S_1 | R_1 \rangle$, $G_2 = \langle S_2 | R_2 \rangle$. WLOG $S_1 \cap S_2 = \emptyset$.

There exist inverse isomorphisms $\phi : G_1 \to G_2$, $\psi : G_2 \to G_1$. $\forall s \in S_1$, choose $w_s \in F(S_2)$ representing $\phi(s)$ in G_2. $\forall t \in S_2$, choose $v_t \in F(S_1)$ representing $\psi(t)$ in G_1.

Take the two subsets of $F(S_1 \cup S_2)$:

$$U_1 = \{s^{-1} w_s : s \in S_1\}, \qquad U_2 = \{t^{-1} v_t : t \in S_2\}.$$

Claim: There exist finitely many Tietze transformations from $\langle S_1 | R_1 \rangle$ to $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_1 \cup U_2 \rangle$.

Tietze transformations

Claim: There exist finitely many Tietze transformations from $\langle S_1 | R_1 \rangle$ to $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_1 \cup U_2 \rangle$.

Proof of claim: Use finitely many (T2) to get from $\langle S_1 | R_1 \rangle$ to $\langle S_1 \cup S_2 | R_1 \cup U_2 \rangle$. There exists an isomorphism

$$\rho : \langle S_1 \cup S_2 | R_1 \cup U_2 \rangle \to \langle S_1 | R_1 \rangle \quad \rho(s) = s, \forall s \in S_1 \quad \rho(t) = v_t, \forall t \in S_2$$

Then $\phi \circ \rho : \langle S_1 \cup S_2 | R_1 \cup U_2 \rangle \to \langle S_2 | R_2 \rangle$ is an isomorphism such that $t \overset{\rho}{\mapsto} v_t \overset{\phi}{\mapsto} t$. Also, $\forall r \in R_2$

$$\phi \circ \rho(r) = r \equiv 1 \text{ in } \langle S_2 | R_2 \rangle \Rightarrow r \in \langle\langle R_1 \cup U_2 \rangle\rangle \Rightarrow R_2 \subseteq \langle\langle R_1 \cup U_2 \rangle\rangle$$

Thus $\langle S_1 \cup S_2 | R_1 \cup U_2 \rangle$ is related to $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_2 \rangle$ by a sequence of (T1) transformations. Also, $\forall s \in S_1$

$$\phi \circ \rho(s) = w_s(t_1 \dots t_k) \quad \phi \circ \rho(w_s) = \phi \circ \rho(w_s(t_1 \dots t_k)) = w_s(t_1 \dots t_k)$$

Hence, $s^{-1} w_s \in \langle\langle R_1 \cup U_2 \rangle\rangle$, which implies that $U_1 \subseteq \langle\langle R_1 \cup U_2 \rangle\rangle$. So we can apply several (T1) to get $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_1 \cup U_2 \rangle$. □

Properties of finite presentability

Proposition

Let G be a group.

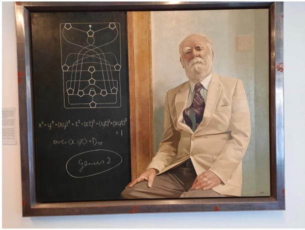
1. *G finitely presented does not imply that a subgroup is finitely presented or that a quotient is finitely presented.*
2. *If H is a finite index subgroup of G then G is finitely presented if and only if H is.*
3. *If $N \trianglelefteq G$ is finitely presented and G/N is finitely presented then G is finitely presented.*

A proof can be found in the notes.

Graham Higman

Remark

G finitely presented does not imply that a subgroup is finitely presented.



Theorem

Every finitely generated recursively presented group can be embedded as a subgroup of some finitely presented group.