Geometric Group Theory

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My favourite quotation

William Thurston:
"Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding."

List of algorithmic problems of M. Dehn

Word problem: Given a finite presentation $G = \langle S | R \rangle$ design an algorithm recognising when $w \in F(S)$ satisfies $w = 1_G$ in G.

Conjugacy problem: Given a finite presentation $G = \langle S | R \rangle$ design an algorithm recognising when $u, v \in F(S)$ represent conjugate elements in G.

Remark

The conjugacy problem implies the word problem.

Isomorphism problem: Given finite presentations $G_i = \langle S_i | R_i \rangle, i = 1, 2$, determine if $G_1 \simeq G_2$.

Triviality problem (a particular case of the isomorphism problem): Given a finite presentation $G = \langle S | R \rangle$ determine if $G \simeq \{1\}$.

Novikov, Boone, Rabin ['56]: All of the above are unsolvable.

Fridman ['60]: There exists a group with solvable word problem, but unsolvable conjugacy problem.

Word and conjugacy problems

Proposition

If the word problem or conjugacy problem is solvable for $G = \langle S|R\rangle$ then it is solvable for any finite $\langle X|Q\rangle = G$.

Proof.

WP: Given $w \in F(X)$ we run simultaneously 2 procedures:

1. List all elements in $\langle\langle Q\rangle\rangle$ (i.e. multiply conjugates $q_i^{w_i}, w_i \in F(X), q_i \in Q$ and transform into reduced word); check if w is among them. If yes, stop and conclude $w = 1$.
2.
   a. List all homomorphisms $\phi : F(X)/\langle\langle Q\rangle\rangle \to F(S)/\langle\langle R\rangle\rangle$ (i.e. enumerate all $|X|$-tuples of words in $F(S)$, then check if each $q \in Q$, rewritten by changing $x \mapsto w_x$, becomes $\equiv 1$ in $F(S)/\langle\langle R\rangle\rangle$). This can be done since the WP for $\langle S|R\rangle$ is solvable.
   b. For each ϕ, check if $\phi(w) \neq 1$ in $F(S)/\langle\langle R\rangle\rangle$. If yes, stop and conclude $w \neq 1$.

Word and conjugacy problems

Proof continued: CP: Given $w, v \in F(X)$, run the following 2 procedures in parallel:

1. a. List all $gvg^{-1}w^{-1}$ in $F(X)$.
   b. Check if $gvg^{-1}w^{-1}$ is among the list of elements in $\langle\langle Q \rangle\rangle$. If yes, stop and conclude: "v, w conjugate".
2. a. List all homomorphisms $\phi : F(X)/\langle\langle Q \rangle\rangle \to F(S)/\langle\langle R \rangle\rangle$.
   b. Check if $\phi(v), \phi(w)$ are not conjugate. If yes, stop and conclude: "v, w not conjugate".

□

Residually finite groups

Idea: Approximate by finite quotients. So we will need enough of those.
NB In what follows, we do not assume finite generatedness.

Lemma

TFAE

1. $$\bigcap_{H \leq_{f.i.} G} H = \{1\}$$
2. *For all non-trivial $g \in G$, there exists $\phi : G \to F$ finite such that $\phi(g) \neq 1$.*
3. *For all $\{g_1, ..., g_n\}$ distinct, there exists $\phi : G \to F$ such that $\phi(g_1), ..., \phi(g_n)$ are distinct. In other words, every finite chunk of the infinite Cayley table of G can be reproduced identically in the Cayley table of a finite quotient.*

Residually finite groups

Proof.

The proof is based on the fact that

$$\bigcap_{H \leq_{f.i.} G} H = \bigcap_{N \trianglelefteq_{f.i.} G} N$$

The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are OK.

And for (1) $\Rightarrow$ (3): $\forall i \neq j$, take $N_{ij} \not\ni g_i g_j^{-1}$ and define

$$N = \bigcap_{i \neq j} N_{ij}$$

and then consider $\phi : G \to G/N$. □

Residually finite groups

Examples

1. *$GL(n, \mathbb{Z})$ is residually finite.* $\forall g \neq \mathrm{id}$:
    - a. *If* $\exists i \neq j$ *such that* $|g_{ij}| \neq 0$, *take* $p > |g_{ij}|$ *and reduce mod* p.
    - b. *If* $\forall i \neq j$, $g_{ij} = 0$, *then* $\exists$ $g_{ii} = -1$. *Reduce mod* 3: $g_{ii} = 2$.
2. *Any finitely generated $G \leq SL(n, \mathbb{Q})$ (or $GL(n, \mathbb{Q})$) is RF.*
3. *$(\mathbb{Q}, +)$ is not RF (and nor is $SL(n, \mathbb{Q})$):*

    if we have some $\phi : \mathbb{Q} \to F$, $\phi(0) = \mathrm{id}$, *take* $g = \phi(1)$, $n = |F|$ *and then* $g = \phi(1) = \phi(\frac{1}{n})^n = \mathrm{id}$.

Theorem (Mal'cev)

Let R be a commutative ring with unity. Any finitely generated $G \leq GL(n, R)$, G is residually finite.

Residually finite groups

Proposition

1. *If G is RF and $H \leq G$ then H is RF.*
2. *If H is a finite index subgroup of G then H is RF if and only if G is RF.*
3. *If two groups G and H are RF then $G \times H$ is RF.*
4. *If $G = H \rtimes K$ where H is finitely generated RF, K is RF, then G is RF.*

Proposition

For all finite or countable X, $F(X)$ is residually finite.

Proof: We have that $F_2 \simeq \langle \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \rangle \leq GL(2, \mathbb{Z})$. And for all X finite or countable, $F(X) \leq F_2$. $\square$