Geometric Group Theory

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Word and conjugacy problems

Proposition

If the word problem is solvable for $G = \langle S|R \rangle$ then it is solvable for any finite $\langle X|Q \rangle = G$.

Proof.

Given $w \in F(X)$ we run simultaneously 2 procedures:

1. List all elements in $\langle\langle Q \rangle\rangle$ (i.e. multiply conjugates $q_i^{w_i}, w_i \in F(X), q_i \in Q$ and transform into reduced word); check if w is among them. If yes, stop and conclude $w = 1$.
2.
   a. List all homomorphisms $\phi : F(X)/\langle\langle Q \rangle\rangle \to F(S)/\langle\langle R \rangle\rangle$ (i.e. enumerate all $|X|$-tuples of words in $F(S)$, then check if each $q \in Q$, rewritten by changing $x \mapsto w_x$, becomes $\equiv 1$ in $F(S)/\langle\langle R \rangle\rangle$). This can be done since the WP for $\langle S|R \rangle$ is solvable.
   b. For each ϕ, check if $\phi(w) \neq 1$ in $F(S)/\langle\langle R \rangle\rangle$. If yes, stop and conclude $w \neq 1$.

Residually finite groups

Theorem

A finitely presented residually finite group has a solvable word problem.

Remark

Note that every finite group has a solvable word problem.

Proof.

Suppose $G = \langle S | R \rangle$. Take $w \in F(S)$. Run simultaneously two procedures:

1. List all the elements in $\langle\langle R \rangle\rangle$ and check if w is among them.
2. List all homomorphisms $\phi : F(S)/\langle\langle R \rangle\rangle \to S_n, n \in \mathbb{N}$, and check if $\phi(w) \neq 1$.

□

Residually finite groups

Definition

G is Hopf if every onto homomorphism $f : G \to G$ is an isomorphism.

Example

Every finite group is Hopf.

Theorem

A finitely generated residually finite group is Hopf.

Residually finite groups

Theorem

A finitely generated residually finite group is Hopf.

Proof.

Assume there exists an onto homomorphism $f : G \to G$ that is not 1-to-1.

Take $g \in \ker f \setminus \{1\}$. There exists $\phi : G \to F$ with $\phi(g) \neq 1$. Construct a sequence

$$g = g_0,\ \ g_1 \in f^{-1}(g_0),\ \ g_2 \in f^{-1}(g_1),\ \ \ldots\ ,\ \ g_n \in f^{-1}(g_{n-1})$$

$\forall n$, $f^n(g_n) = g$ and $f^k(g_n) = 1$ for all $k > n$. Hence, for all $n > \ell$, $\phi \circ f^n(g_\ell) = 1$ and $\phi \circ f^n(g_n) \neq 1$. So the homomorphisms $\phi \circ f^n$ are pairwise distinct. But this contradicts $\mathrm{Hom}(G, F) \leq |F|^{|S|}$. □

Residually finite groups

Theorem

A finitely generated residually finite group is Hopf.

Corollary

If $F(X) = \langle A \rangle$ and $|A| = |X| = n < \infty$, then $F(X) \simeq F(A)$. (i.e. A freely generates $F(X)$ i.e. A is a free basis for $F(X)$).

Proof.

A bijection $X \to A$ extends to $X \to F(A)$ which extends to an onto homomorphism $F(X) \to F(A)$. By Universal Property, we have a second onto homomorphism, hence an onto hom. $F(X) \to F(A) \to F(X)$. Since $F(X)$ is Hopf, the latter hom. is an isomorphism, hence all are. □

Residually finite groups. Simple groups

Theorem

A finitely generated residually finite group is Hopf.

The assumption finitely generated cannot be dropped from the theorem.

Example

- *Consider* X, Y *countable.*
- *There exists* $f : X \to Y$ *onto and not injective.*
- f *extends uniquely to an onto group homomorphism* $F(X) \to F(Y)$.

At the other extreme, we have simple groups.

Definition

G is simple if the only normal subgroups are $\{1\}$ and G.

Simple groups

Example

$\mathbb{Z}/p\mathbb{Z}$, A_n, A_∞, $PSL(n, \mathbb{Q})$, infinite f.g. due to Higman, Thompson, Olshanskii, Burger-Mozes.

Theorem

A finitely presented simple group has solvable word problem.

Proof.

Let $w \in F(S)$. Since G is simple, if $w \neq 1$ in G then $G = \langle\langle w \rangle\rangle$ and hence $\langle\langle \{w\} \cup R \rangle\rangle = F(S)$.

Two procedures:

1. Enumerate $\langle\langle R \rangle\rangle$. Check if w appears.
2. Enumerate $\langle\langle \{w\} \cup R \rangle\rangle$. Check if every $s \in S$ appears.

□

Graphs and Cayley graphs

A main method of investigation is to endow an infinite group with a geometry compatible with its algebraic structure, i.e. invariant by multiplication. This can easily be done for finitely generated groups via Cayley graphs.

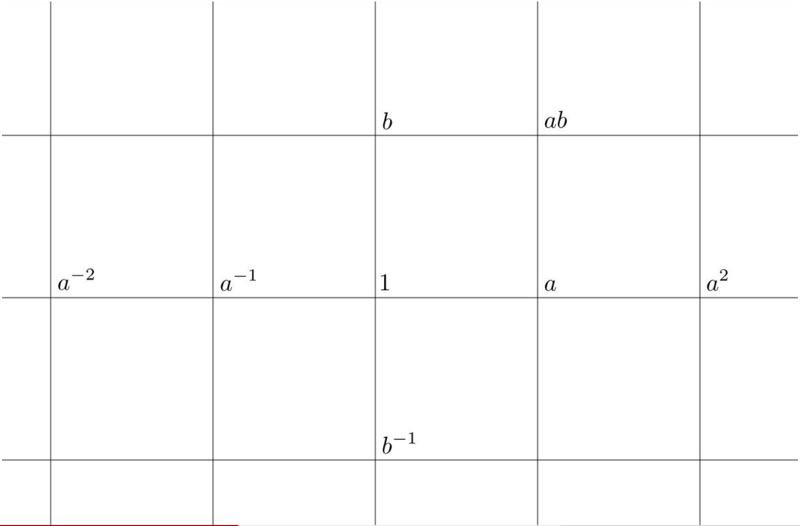
Given a countable group G and a subset S such that $S^{-1} = S$, the Cayley graph of G with respect to S, denoted $\Gamma(S, G)$, is a directed/oriented graph with

- set of vertices G;
- set of oriented edges $\{(g, gs) : g \in G, s \in S\}$;

We denote an edge $[g, gs]$. The underlying non-oriented graph is denoted $\hat{\Gamma}(S, G)$.

Examples of Cayley graphs

1. $\mathbb{Z}^2$ with $S = \{(\pm 1, 0), (0, \pm 1)\}$

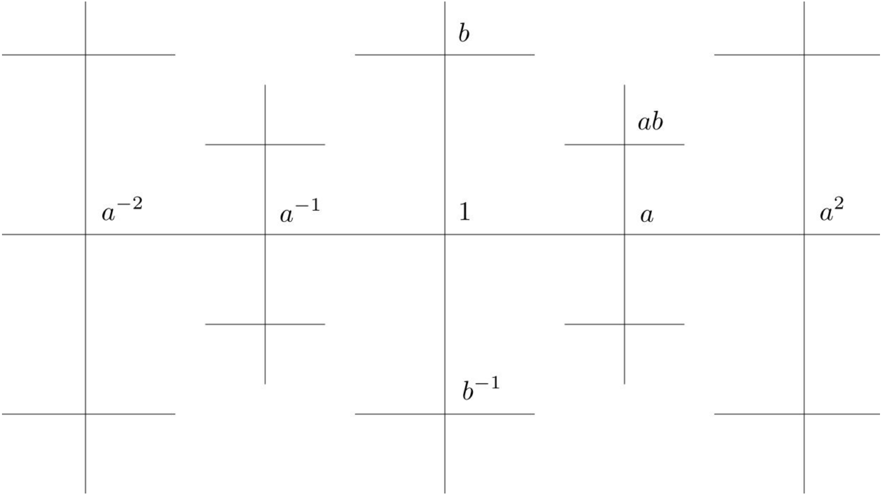


Examples of Cayley graphs

2. $\mathbb{Z}^2$ with $S = \{(\pm 1, 0), \pm(1,1)\}$

Examples of Cayley graphs

3. $F_2 = F(\{a, b\})$ with $S = \{a^{\pm 1}, b^{\pm 1}\}$



Examples of Cayley graphs: the integer Heisenberg group

The Integer Heisenberg group:

$$H_{2n+1}(\mathbb{Z}) := \langle x_1, \dots, x_n, y_1, \dots, y_n, z\,;$$

$$[x_i, z] = 1, [y_j, z] = 1, [x_i, x_j] = 1, [y_i, y_j] = 1, [x_i, y_j] = z^{\delta_{ij}}, 1 \leqslant i, j \leqslant n\rangle.$$

$$H_{2n+1}(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & \dots & \dots & x_n & z \\ 0 & 1 & 0 & \dots & \dots & 0 & y_n \\ 0 & 0 & 1 & \dots & \dots & 0 & y_{n-1} \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 & y_2 \\ 0 & 0 & \dots & \dots & 0 & 1 & y_1 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \; ; \; x_i, y_j, z \in \mathbb{Z} \right\}$$

Examples of Cayley graphs: the Integer Heisenberg group

5. $H_3(\mathbb{Z}) := \langle x, y, z \mid [x, z] = 1, [y, z] = 1, [x, y] = z \rangle$.

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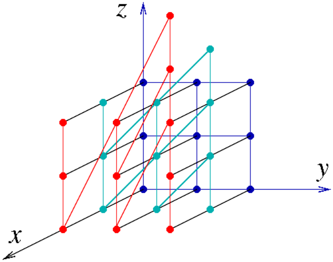

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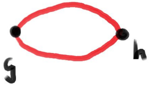
Particular features of Cayley graphs

1. No monogons (edges of the form $[g, g]$) if $1 \notin S$.

2. No digons if, when $s = s^{-1}$, we do not list both s and s^{-1} in S (i.e. no repetitions in S).



   In other words, this is a simplicial graph.

3. $\Gamma(S, G)$ is connected (i.e. any two vertices can be connected by an edge path) if and only if $G = \langle S \rangle$.
4.
   a. $\Gamma(S, G)$ is regular: the valency/degree of every vertex is $|S|$.
   b. $\Gamma(S, G)$ is moreover locally finite if and only if $|S| < \infty$.

Particular features of Cayley graphs

5. If $\Gamma(S, G)$ is infinite then it always contains a bi-infinite geodesic.

6. $\Gamma(S, G)$ always contains a cycle (in fact plenty) with one exception: $\Gamma(S, G)$ does not contain a cycle (i.e. it is a simplicial tree) $\iff$ $S = X \sqcup X^{-1}$ and $G = F(X)$.