Representation theory of semisimple Lie algebras (Part C)

Number of lectures: 16 HT

Course description: Level: M-level Method of Assessment: Written examination. Weight: Unit (OSS paper code ??).

Recommended prerequisites: Past attendance at Lie algebras is recommended, but not required. Past attendance at Introduction to Representation Theory is recommended as well, but not required.

Overview: The representation theory of semisimple Lie algebras plays a central role in modern mathematics with motivation coming from many areas of mathematics and physics, for example, the Langlands program. The methods involved in the theory are diverse and include remarkable interactions with algebraic geometry, as in the proofs of the Kazhdan-Lusztig and Jantzen conjectures.

The course will cover the basics of finite dimensional representations of semisimple Lie algebras (e.g., the Cartan-Weyl highest weight classification) in the framework of the larger Bernstein-Gelfand-Gelfand category $\mathcal{O}$.

Learning Outcomes: The students will have developed a comprehensive understanding of the basic concepts and modern methods in the representation theory of semisimple Lie algebras, including the classification of finite dimensional modules, the classification of objects in category $\mathcal{O}$, character formulas, Lie algebra cohomology and resolutions of finite dimensional modules.

Synopsis:

- Universal enveloping algebra of a Lie algebra, Poincaré-Birkhoff-Witt theorem, basic definitions and properties of representations of Lie algebras, tensor products.
- The example of $\mathfrak{sl}(2)$: finite dimensional modules, highest weights.
- Category $\mathcal{O}$: Verma modules, highest weight modules, infinitesimal characters and Harish-Chandra's isomorphism, formal characters, contravariant (Shapovalov) forms.
- Finite dimensional modules of a semisimple Lie algebra: the Cartan-Weyl classification, Weyl character formula, dimension formula, Kostant's multiplicity formula, examples.
- Homological algebra: Lie algebra cohomology, Bernstein-Gelfand-Gelfand resolution of finite dimensional modules, Ext groups in category $\mathcal{O}$.
- Topics: applications, Bott's dimension formula for Lie algebra cohomology groups, characters of the symmetric group (via Zelevinsky's application of the BGG resolution to Schur-Weyl duality).

Reading List:

- Course Lecture Notes.
- J. Bernstein, Lectures on Lie algebras, in "Representation Theory, Complex Analysis, and Integral Geometry", Springer 2012.

Further reading:

- J. Humphreys, Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$, AMS 2008.
- J. Humphreys, Introduction to Lie algebras and representation theory, Springer 1997.
- W. Fulton, J. Harris, Representation Theory, Springer 1991.