Morphisms of Affine Schemes

Brian Reinhart

February 5, 2025

Definition 0.1. An **affine scheme** is a locally ringed space which is isomorphic to $\operatorname{Spec} R$ for some ring R.

Affine schemes form a category! It's tempting to say "of course they do, it comes from the category of sheaves", but remember that sheaves only form a category when they're over the same topological space. We need a different notion of morphism.

Specifically, given a homomorphism of rings $\phi : R \to S$, we want this to correspond, ideally uniquely, to a morphism $\operatorname{Spec}(\phi) : \operatorname{Spec} S \to \operatorname{Spec} R$ of schemes, coming from the map ϕ^* of topological spaces. Can we define such morphisms purely in the language of sheaves?

Lemma 0.2. *If $\alpha = \operatorname{Spec}(\phi) : \operatorname{Spec} S \to \operatorname{Spec} R$, then $\alpha^{-1}(D_f) = D_{\phi(f)}$.*

So we can sort of recover ϕ using from $\operatorname{Spec} \phi$. The proof of this is straightforward.

Denote $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$.

Proposition 0.3. *There is a morphism of sheaves $\phi^\# : \mathcal{O}_X \to \alpha_* \mathcal{O}_Y$ on X such that $\phi^\#_X : \mathcal{O}_X(X) \cong R \to \alpha_* \mathcal{O}_Y(X) \cong S$ is equal to ϕ.*

Proof. We can just define $\phi^\#$ on the basis of D_f and then use that to determine behavior on stalks, and thus on every open set. It should map $\mathcal{O}_X(D_f)$ to $\alpha_* \mathcal{O}_Y(D_f)$, which is just $\mathcal{O}_Y(\alpha^{-1} D_f) = \mathcal{O}_Y(D_{\phi(f)})$. But we have a map which works for this: ϕ itself, but extended to take $\phi(1/f) = 1/\phi(f)$! Just need to do the simple check that this is compatible with restriction maps. □

Proposition 0.4. *For each $\mathfrak{p} \in Y$, $\phi^\#_{\mathfrak{p}} : \mathcal{O}_{X,\alpha(\mathfrak{p})} \to \mathcal{O}_{Y,\mathfrak{p}}$ is a **local morphism**: a ring homomorphism of local rings for which the image of the unique maximal ideal in the domain lies in the unique maximal ideal in the codomain.*

Proof. $\mathcal{O}_{X,\alpha(\mathfrak{p})} \cong R_{\phi^{-1}(\mathfrak{p})}$ and $\mathcal{O}_{Y,\mathfrak{p}} \cong S_{\mathfrak{p}}$. Basically by definition, $\phi^\#$ takes $\phi^{-1}(\mathfrak{p})$ in each section of $\mathcal{O}_X$ to a subset of $\mathfrak{p}$ in each section of $\mathcal{O}_Y$, and thus also does so in the product and in the direct limit. □

The upshot: we have a notion of a morphism of locally ringed spaces.

Definition 0.5. Given two locally ringed spaces X, Y, a **morphism** from Y to X is a pair of morphisms

- $\alpha : Y \to X$ of topological spaces.
- $\alpha^\# : \mathcal{O}_X \to \alpha_* \mathcal{O}_Y$ of sheaves on X.

such that $\alpha^\#$ is a local morphism on stalks.

Spec turns out to be a fully faithful contravariant functor from the category of rings to the category of locally ringed spaces (see Ritter 1.13 for proof). Aff ,the category of affine schemes, is exactly the locally ringed spaces that are isomorphic to something in the image of this functor. Thus we have

$$\operatorname{Spec} : \text{Rings}^{op} \to \text{Aff}$$

is an equivalence of categories.