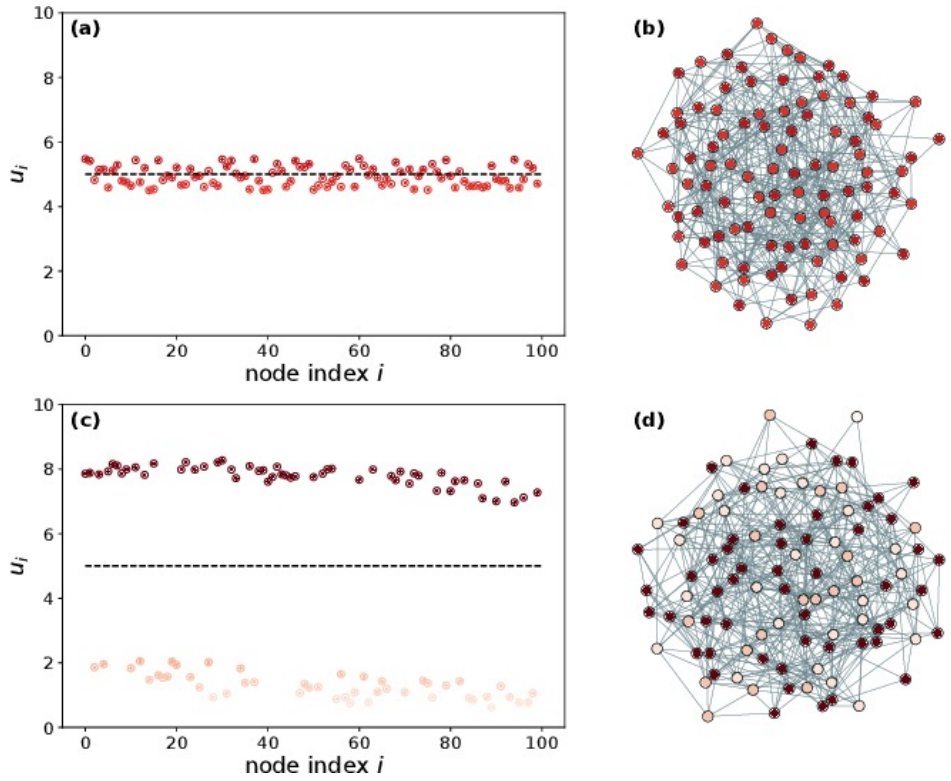
Networks C5.4
Lecture 9



6 DYNAMICS ON NETWORKS

One of the main motivations for identifying modular structures in networks is that they provide a simplified, coarse-grained description for the system structure.

Think for instance of a social network, in which we might be able to decompose the system into groups of people such as circles of friends. We may then represent the system in terms of the interactions between these different groups of people, thereby reducing the complexity of our description. The hope is to not only arrive at a more compact structural description, but that the found modules can be interpreted as the 'building blocks' of the system with a functional meaning.

In general this functionality is expressed in the ways by which dynamics is constrained by the underlying structure. Think of flows of passengers in the underground, flows of ideas in citations networks, or flow of information in the social network example. To properly understand such systems, we indeed to consider the dynamics that acts on top of an underlying structure structure. In this section, we will provide an overview of the interplay between structure and dynamics in complex networks by considering (linear) consensus dynamics. First, we describe dynamics with a separation of time-scales and discuss how such a time-scale separation can be a direct consequence of the network structure. Second, we discuss how the presence of particular symmetries in a network can give rise to in- variant subspaces in the dynamics that can be precisely described by graph partitions.

Here is a short summary of the notations used in this section. For simplicity, in the following we consider mainly undirected, connected graphs with n vertices and m edges, as usual.

Our ideas extend to directed graphs, however, which we will outline as we go along. The topology of a graph is encoded in the weighted (non-negative) adjacency matrix $A \in \mathbb{R}^{n \times n}$, where the weight of the edge between vertex i and vertex j is given by A_{ij}. Note that $A = A^T$ for an undirected graph.

The weighted out-degrees (or strengths) of the nodes are given by the vector out-degrees $\mathbf{d} = A.\hat{1}$, where $\hat{1}$ denotes the n-vector of ones. For any vector $\mathbf{x} = (x_1, \ldots, x_n)^T$, we define diag($\mathbf{x}$) to be the diagonal matrix X with elements $X_{ii} = x_i$ and zero otherwise. We thus define the diagonal matrix of degrees as $D =$diag(d).

Recall also that combinatorial graph Laplacian is defined as $L = D - A$. It is symmetric (self adjoint) and positive semi-definite, with a simple zero eigenvalue when the graph is connected.

6.1 Consensus dynamics

Consensus has been one of the most popular and well- studied dynamics on networks. This is due to both its analytic tractability as well as its simplicity in approximating several fundamental behaviours.

For instance, in socio-economic domains consensus provides a model for opinion formation in a society of individuals. For engineering systems, it has been considered as a basic building block for an efficient distributed computation of global functions in networks of sensors, robots, or other agents. To define a standard consensus dynamics, con- sider a given connected network of n nodes and adjacency matrix A. Let us endow each node with a scalar state variable $x_i \in \mathbb{R}$, which is a function of time t. Let $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^T$.

Suppose $v_i$and v_j are neighbours. Then v_i sees a difference of $(x_j - x_i)$ in their states. If this is positive (respectively negative) then v_i should increase (respectively decrease) its states so as to close the gap.

Linear System

So we have a full solution

The (average) **consensus dynamics** on such a network is then defined by the autonomous differential equation:

$$\dot{\mathbf{x}} = -L\mathbf{x}. \tag{10}$$

Note that in coordinate form this simply amounts to $\dot{x}_i = \sum_j A_{ij}(x_j - x_i)$, that is, the state of each vertex adjusts so that the difference to its neighbours is reduced. The name of these dynamics de-rives from the fact that for any given initial system state $\mathbf{x}_0 = \mathbf{x}(0)$, the differential equation above will drive the state to a global 'consensus state' in which the state variables of all nodes are equal.

Exercise. Write a code to simulate consensus dynamics on a network, and verify that the dynamics asymptotically converges towards the state $x^*\hat{1}$.

Since $\hat{1}^T.L = 0$, $\hat{1}$ is an eigenvector of L with zero eigenvalue, we see that

$$\hat{1}^T.\dot{\mathbf{x}} = 0 \rightarrow \hat{1}^T.\mathbf{x} = \text{ constant.}$$

Mathematically, this means that $x_i \rightarrow x^*$ for all i, as $t \rightarrow \infty$, where $x^* = \hat{1}^T.\mathbf{x}_0/n$ is the arithmetic average of the initial node states. Intuitively, this dynamics may be interpreted as an opinion formation process on a network of agents, who will in the absence of further inputs eventually agree on the same value: namely, the average opinion of their initial states.

The rate of convergence is limited by the second smallest eigenvalue of L, the Fieldler eigenvalue, λ_F say. with

$$\mathbf{x}(t) = x^*\hat{1} + O(e^{-\lambda_F t}).$$

To see this simply expand $\mathbf{x}(t)$ out in terms of the orthonormal eigenvectors of L, and observe that the resulting scalar equations decouple. Note that, of course, if the network is NOT strongly connected, so 0 is not a simple eigenvalue for L, then we can solve this equation on each connected component independently, whence $\lambda_F > 0$.

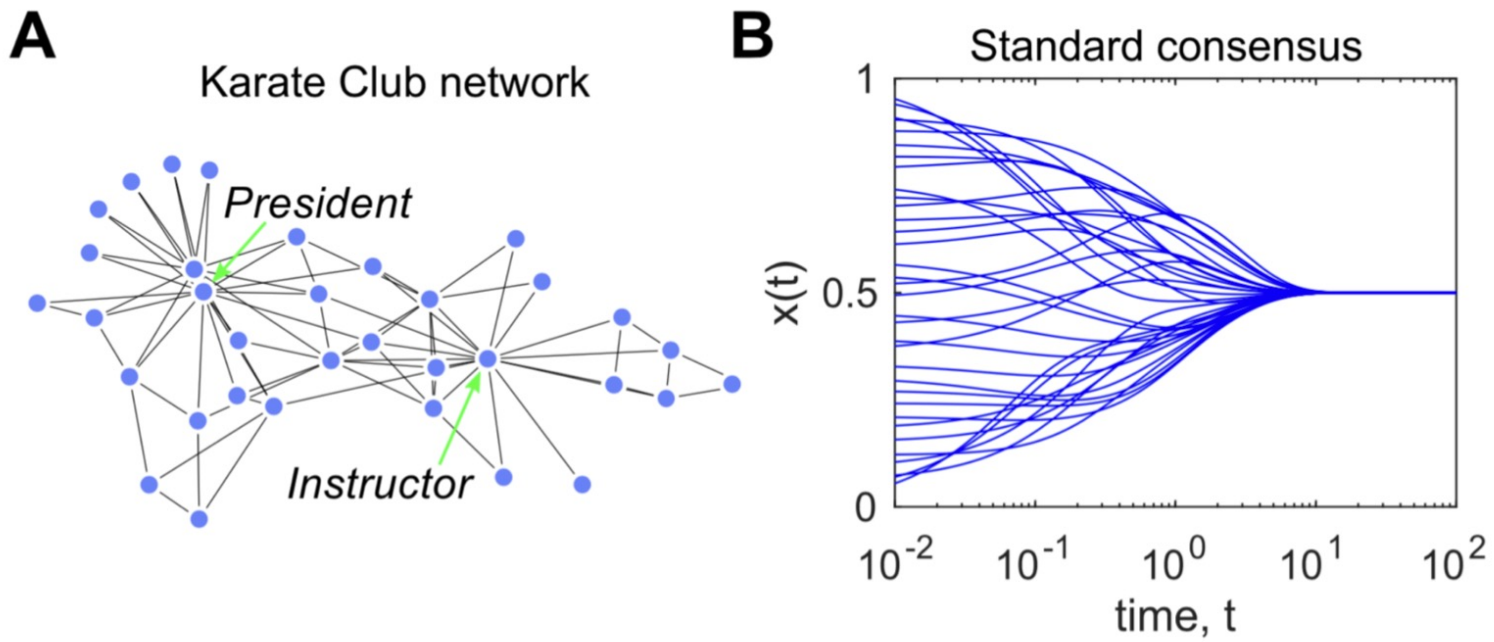


Figure 19: Illustration of a consensus dynamics on the small, often studied, Karate Club network (https://en.wikipedia.org/wiki/Zachary%27s_karate_club), originally analysed by Zachary. A: the network. B: Consensus dynamics on the Karate club network starting from a random initial condition. As discussed in the text, as time progresses the states of the individual nodes become more and more aligned, and eventually reach a consensus value, equal to the arithmetic average of the initial condition.

6.2 Time-scale separation in dynamical systems

Before discussing time-scale separation in the context of a dynamical process acting on a network, let us explain the concept of time-scale separation with a generic example first.

Consider the following simple dynamical system:

$$\frac{dx}{dt} = f(x, y),$$

$$\epsilon^{-1}\frac{dy}{dt} = g(x, y),$$

where $\epsilon << 1$ is a small positive constant. Note that, the above implies that $x(t)$ changes much more rapidly than $y(t)$. Indeed, dy/dt will be proportional to $\epsilon g(x, y)$, which is small by construction. Alternatively, simply define a new, slow time variable $\tau := \epsilon t$ and note that the above can be written as

$$\frac{dx}{dt} = f(x, y),$$

$$\frac{dy}{d\tau} = g(x, y).$$

There is a separation of time-scales in the dynamics, where y evolves according to the 'slow' timescale τ, and x according to the much faster timescale, t

This time-scale separation can be exploited for the analysis of a system in various ways. On the one hand, if we are mainly interested in the short term (fast) behaviour of the system above, we may effectively treat y as a fixed parameter and ignore its time evolution, leading to an effective one-dimensional system description. Indeed for the so called *singular perturbation*, $\epsilon \to 0$, y and x will effectively be decoupled.

On the other hand, if we are mainly interested in the long term behaviour of the system, then it is y we are most interested in. Let us assume, that x(t) converges to some fixed point $x^*(y)$ on the fast timescale, t, as a function of y which is effectively constant on that timescale. Then on the slow time scale we have

$$\frac{dy}{d\tau} = g(x^*(y), y).$$

Using this simplification will, of course, lead to some errors when comparing to the actual time-evolution of y, especially for an initial transient period when f is a long way from zero. However, it allows us again to focus on a simpler one-dimensional system, facilitating a simpler analysis.

This general idea is the basis for center manifold theory and the asymptotic method of multiple timescales [33]. To summarise, the separation of time-scales acts in such a way that it almost decouples the system in two different regimes.

6.3 Time-scale separation in networks with consensus dynamics

Let us now discuss how the above ideas can be trans-lated into the context of networks on which a diffusion or consensus dynamics is acting. For simplicity we will de- scribe the results here in the context of consensus, though translating these ideas to diffusion processes is straight- forward.

For any initial condition $\mathbf{x}_0$, standard linear systems theory tells us that the solution to 10 is given by

$$\mathbf{x}(t) = \exp(-Lt)\mathbf{x}_0,$$

where $\exp(\Delta)$ denotes the matrix exponential. Writing the solution in this way disguises however the time-scales present in the evolution of x as these gets mixed via the network interactions. In order to reveal the characteristic time-scales present in the system we can make use of a spectral decomposition of L. Let us denote the eigenvectors of the Laplacian by $\mathbf{v}_i$, i.e., $L\mathbf{v}_i = \lambda_i \mathbf{v}_i$, and assume that we have ordered the eigenvalues (and associated eigenvectors) in increasing order $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

We may now decompose the Laplacian via

$$L = \sum_i \lambda_i \mathbf{v}_i \mathbf{v}_i^T,$$

and write

$$\mathbf{x}(t) = \sum_i \exp(-\lambda_i t)\mathbf{v}_i \mathbf{v}_i^T \mathbf{x}_0.$$

Linear System

So we have a full solution

$$\mathbf{x}(t) = \sum_i \exp(-\lambda_i t)\mathbf{v}_i\mathbf{v}_i^T\mathbf{x}_0.$$

In this format the time-scales of the process become apparent. They are dictated by the eigenvalues of the Laplacian matrix: each eigenvector (or eigenmode) decays according to a characteristic time-scale $\tau_i = 1/\lambda_i$. Hence, if there are large differences between the eigenvalues, we will have a time-scale separation. More precisely assume that there is a group of k small eigenvalues $\{0, ..., \lambda_k\}$, which are well separated from the remaining eigenvalues in the sense that $\lambda_k << \lambda_{k+1}$. Then after some time, $\tau_0 \sim 1/\lambda_{k+1}$, the eigenmodes associated with teh larger eigenvalues become negligible and the system can be effectively described by the k smallest eigenmodes. More technically, the k first eigenvectors form a dominant invariant subspace of the dynamics.

The main point of the discussion above is that if there is a separation of time-scales, there exists a lower dimensional description of the dynamics on the network after a specific time-scale τ_0. A natural question is thus how this time-scale separation and the lower-dimensional description of our dynamics is related to the network structure.

As an example let us consider a network composed out of k modules, only weakly coupled to the other. To simplify our exposition let us consider the case of a graph with k modules, whose adjacency matrix is dominated by k blocks, corresponding to the k modules, and can be written as:

$$A = A_{\text{structure}} + A_{\text{random}} = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix} + A_{\text{random}}.$$

Here A_{random} may be interpreted as a realization from an Erdös-Rényi random graph, with unstructured, sparse connectivity (just a "noise term"); the A_i are the adjacency (sub-)matrices of the individual clusters which have much higher internal connectivity.

How does the structure present affect the spectrum and the eigenvalues of the cor- responding Laplacian $L = L_{\text{structure}} + L_{\text{random}}$?

Let us first consider the case where $L_{\text{random}} = 0$, so the graph consists of k disconnected components. Then we will have $\lambda = 0$ with algebraic (and geometric) multiplicity k, and the associated eigenspace can be spanned by eigenvectors of the form $\mathbf{c}(j)_i = 1$ if vertex i is in component j and zero otherwise.

To gain insight into the case where $L_{\text{random}} \neq 0$, we can appeal to matrix perturbation theory and random matrix theory, respectively. For a network of the form above, the Davis-Kahan theorem provides bounds on the (angular) distance between the subspace Y spanned by $\{\mathbf{c}(j)\}$, and the subspace Y' spanned by the corresponding eigenvectors of L associated with the smallest eigenvalues. On an intuitive level, the Davis-Kahan theorem states that if the noise level is not too high, then $Y \approx Y'$. The implication is that the dominant invariant subspaces will be commensurate with the structural decomposition of the network in terms of the block-vectors. Hence the long-term dynamics will directly reflect the structural decomposition of the network. In other words, the time scale separation in such a networked system takes an intuitive meaning: quasi-consensus is reached more quickly within each block, while global consensus is only reached on a longer time scale.

To illustrate the above discussion let us consider here a numerical example. The network displayed in Figure 20A consists of 3 groups with 100 nodes each, and is structured as outlined in our discussion above. As can be seen clearly in Figure 20B the dynamics becomes effectively low dimensional after around $t = 0.05$ and can be well approximated by the dominant eigenmodes.

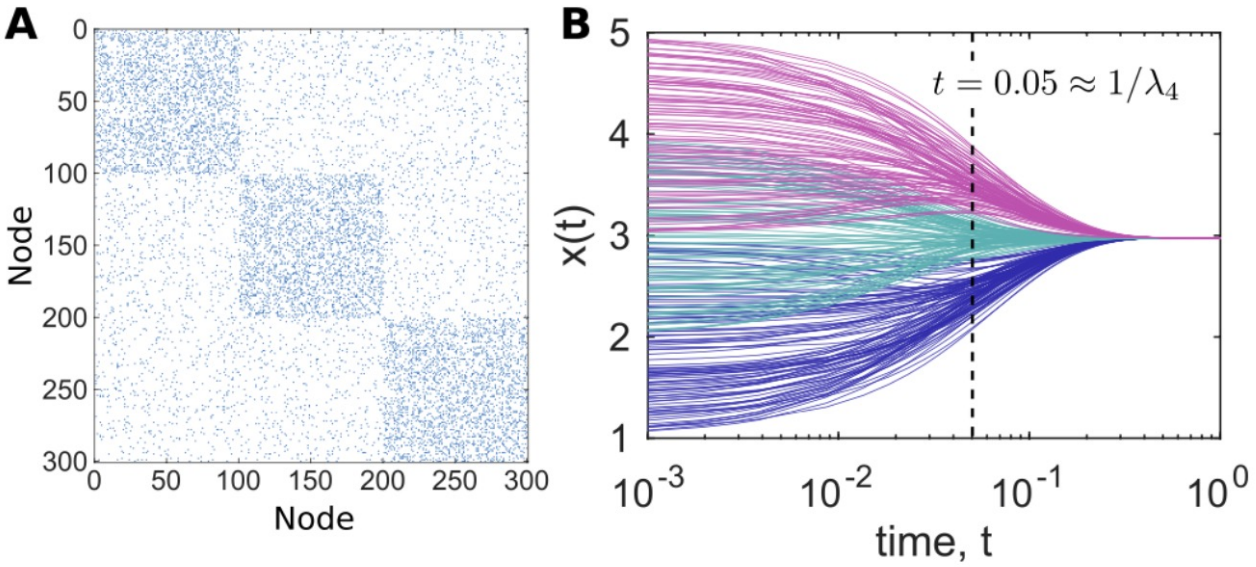


Figure 20: llustration of a consensus dynamics on a structured network. A: Adjacency matrix (unweighted) of a structured network with 3 modules/groups, as discussed in the text. B: Consensus dynamics on this network displays a time-scale separation until around $t = 0.05$, approximate consensus is reached within each group (groups indicated by colour); then a consensus is reached between the groups. Note that for the shown network $\lambda_4 = 18$, in good agreement with our discussion above.

What about nonlinearity?

Activator-inhibitor systems have had an impact within mathematical models where a uniformity equilibrium across a population of individual systems becomes destabilised by the very act of simple passive coupling between them. Such Turing instabilities can sometimes seem counter intuitive on first sight.

Consider a population of n identical individuals, each described by a set of m real attitude state variables, that are continuous functions of time t. Let $\mathbf{x}_i(t) \in \mathbb{R}^m$ denote the ith individual's attitudinal state. Let $A(t)$ denote the adjacency matrix for the communication network, as above in (50). Then consider

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + D_f \sum_{j=1}^{n} A_{ij} \left(\mathbf{x}_j - \mathbf{x}_i\right) \quad i = 1, \ldots, n. \tag{16}$$

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + D_f \sum_{j=1}^{n} A_{ij} \left(\mathbf{x}_j - \mathbf{x}_i\right) \quad i = 1, \ldots, n. \tag{16}$$

Here $\mathbf{f}$ is a given smooth field over $\mathbb{R}^m$, drawn from a class of activator-inhibitor systems (see the exercise below for an example), and is such that $\mathbf{f}(\mathbf{x}^*) = 0$, for some, $\mathbf{x}^*$, and the Jacobian there, $\mathbf{df}(\mathbf{x}^*)$, is a stability matrix, that is, all of its eigenvalues have negative real parts. Here D_f is a real diagonal, non-negative matrix containing the maximal transmission coefficients (diffusion rates) for the corresponding attitudinal variables between adjacent neighbours. Let $\mathbf{X}(t)$ denote the $m \times n$ matrix with ith column given by $\mathbf{x}_i(t)$, and $\mathbf{F}(\mathbf{X})$ be the $m \times n$ matrix with ith column given by $\mathbf{f}(\mathbf{x}_i(t))$. Then (16) may be written as

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}) - D_f \mathbf{X} \Delta. \tag{17}$$

Here, as before, $\Delta(t)$ denotes the graph Laplacian for A , given by $\Delta = \mathrm{D} - A$, where D is the diagonal matrix containing the degrees of the vertices. This system has an equilibrium at $\mathbf{X} = \mathbf{X}^*$ say, where the ith column of $\mathbf{X}^*$ is given by $\mathbf{x}^*$ for all $i = 1, \ldots, n$.

$0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be the eigenvalues of Δ. Then we will show that $\mathbf{X}^*$ is asymptotically stable only if all n matrices, $\mathbf{df}(\mathbf{x}^*) - D_f\lambda_i$, are simultaneously stability matrices; and conversely it is unstable in the ith mode of Δ if $\mathbf{df}(\mathbf{x}^*) - D_f\lambda_i$ has an eigenvalue with positive real part.

To see this consider the linearisation of (17) at $\mathbf{X}^*$, and its behaviour in the ith mode $(i = 1, ..., n)$. We write $\mathbf{X}(t) = \mathbf{X}^* + \mathbf{y}(t)\mathbf{v}_i^T$ where $\mathbf{y}(t) \in \mathbb{R}^m$ is very small, and $\mathbf{v} \in \mathbb{R}^n$ is the ith eigenvector of Δ. Then, retaining only linear terms in the small perturbation, $\mathbf{y}(t)$, we have

$$\dot{\mathbf{y}}\mathbf{v}_i^T = \mathbf{df}(\mathbf{x}^*)\mathbf{y}\mathbf{v}_i^T - D_f\lambda_i\mathbf{y}\mathbf{v}_i^T,$$

so that

$$\dot{\mathbf{y}} = (\mathbf{df}(\mathbf{x}^*) - D_f\lambda_i)\mathbf{y}.$$

Thus the perturbation in the ith mode will tend to zero if and only if $\mathbf{df}(\mathbf{x}^*) - D_f\lambda_i$ is a stability matrix.

Instability for some interval values of λ values = “Turing Instability”

Consider the spectrum of $\mathbf{df}(\mathbf{x}^*) - D_f\lambda$ as a function of λ. If λ is very small then this is dominated by the stability of the uncoupled system, $\mathbf{df}(\mathbf{x}^*)$. If λ is large then this is again a stability matrix, since D is positive definite.

The situation, dependent on some collusion between choices of D_f and $\mathbf{df}(\mathbf{x}^*)$, where there is a **window of instability** for an intermediate range of λ in which at least one of the eigenvalues has positive real part, is known as a Turing instability. It occurs in a very wide class of activator-inhibitor systems.

The Schnackenberg system is a two variable activator inhibitor system, that, using Murray's variables [107, 61], is given by

$$\dot{x_1} = f_1(x_1, x_2) = p - x_1 x_2^2, \;\; \dot{x_2} = f_2(x_1, x_2) = q - x_2 + x_1 x_2^2.$$

Here $p > q > 0$ are constants such that $(p+q)^3 > p - q$. Show that $\mathbf{x}^* = (x_1^*, x_2^*) = (p/(p+q)^2, p+q)$ is a rest point. Linearise the system about that point and show that the trace of the linearisation, $\mathbf{df}(\mathbf{x}^*)$, is negative, while its determinant is positive. Hence the equilibrium is asymptotically stable. Now suppose dispersion is added so that the systems is modified:

$$\dot{x_1} = f_1(x_1, x_2) + d_1(x_1^* - x_1), \;\; \dot{x_2} = f_2(x_1, x_2) + d_2(x_2^* - x_2),$$

for positive constants d_1 and d_2. The linearisation of this system about the same rest point, $\mathbf{x}^*$ yields the matrix $\mathbf{df}(\mathbf{x}^*) - D_f$, where $D_f = \text{diag}\,(d_1, d_2)$. This is our equivalent of the matrix $(\mathbf{df}(\mathbf{x}^*) - D_f \lambda_i)$, introduced above, for the positive eigenvalues of the Laplacian. The sum of the eigenvalues of this system remains negative, and equal to the trace of this matrix. Show that it is possible for the determinant to become zero. Hence the equilibrium can loose stability, specifically, provided that $d_2 < (p-q)/(p+q)$ and d_1 is large enough. State variable x_1 is the inhibitor while x_2 is the activator. If dispersion prevents a rising in x_1 above equilibrium from opposing a rising in x_2, then instability follows.

Answer of exercise 26

First show that $f_1 = f_2 = 0$ at the rest point (x_1^*, x_2^*). Then produce the linear system for small perturbations about this: for example, write $x_i(t) = x_i^* + y_i(t)$, substitute in and retain only the linear terms in the y_is. We have

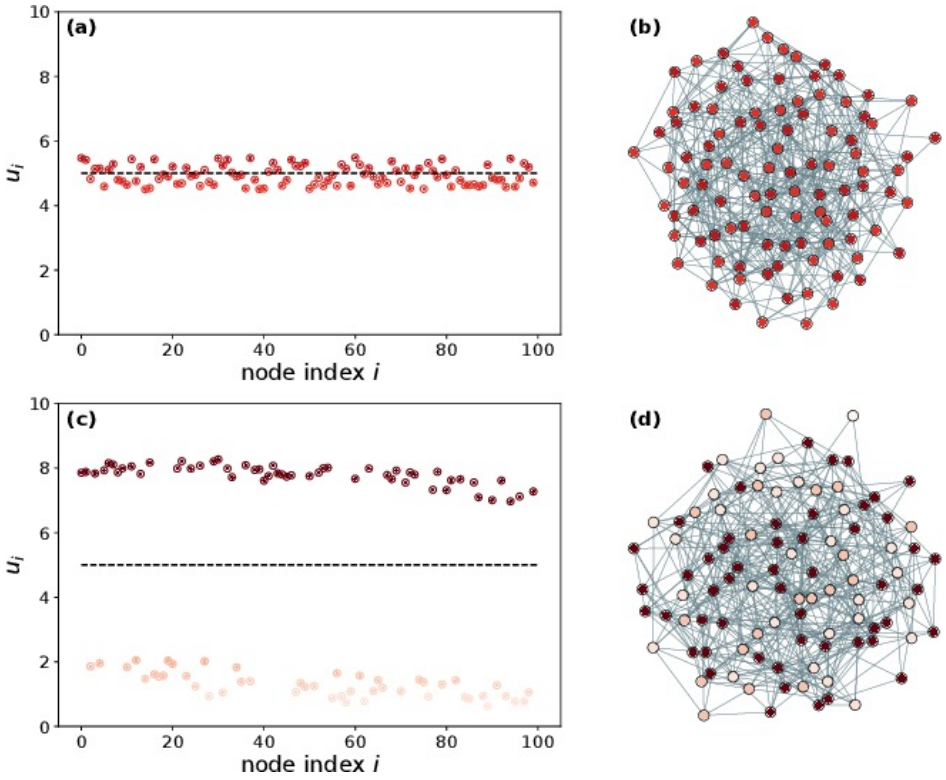
$$\mathbf{df}(\mathbf{x}^*) = \begin{pmatrix} -(p+q)^2 & -2p/(p+q) \\ (p+q)^2 & (p-q)/(p+q) \end{pmatrix}.$$

Thus $\det \mathbf{df}(\mathbf{x}^*) = (p+q)^2 > 0$, and trace $\mathbf{df}(\mathbf{x}^*) = (p-q-(p+q)^3)/(p+q) < 0$.

Now we consider the spectrum of $\mathbf{df}(\mathbf{x}^*) - \mathrm{diag}(d_1, d_2)$. The result follows by examining when the determinant of this matrix become zero and hence there is a zero eigenvalue, and loss of stability as d_1 increases further. See also [61].

So what happens?

- The “common” stable rest point, where **all vertices** are at the resting equilibrium, only exists while no eigenvalues lie in the window of instability
- It loses stability if the Laplacian has at least one eigenvalue inside the window of instability
- Vertices will divide into two subsets according to the elements sign of the corresponding unstable eigenvector and separate equilibria will be attained on each subset
- This is called “pattern formation”
- Usually observed in the analysis of reaction diffusion PDEs in Mathematical Biology



Mimar, Sayat et al. “Turing patterns mediated by network topology in homogeneous active systems.” *Physical review. E* 99 6-1 (2019): 062303 .