Geometric Group Theory

Cornelia Druțu

University of Oxford

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Some inspirational quotations

George Polya: "Where should I start? Start from the statement of the problem. ... What can I do? Visualize the problem as a whole as clearly and as vividly as you can. ... What can I gain by doing so? You should understand the problem, familiarize yourself with it, impress its purpose on your mind."

Th. Bröcker and K. Jänich, "Introduction to differential topology" (p.25) "Having thus refreshed ourselves in the oasis of a proof, we now turn again into the desert of definitions."

Cayley Graphs

From now on, assume that S is a finite generating set (with no repetitions), $1 \notin S$, $S = S^{-1}$. We endow $\Gamma(S, G)$ with a metric d_S:

- each edge has length 1;
- $d_S(x, g)$ is the length of a shortest path from x to g.

Proposition

The action of G on its Cayley graph is an action by isometries. The action is free on the vertices. It is free on the whole Cayley graph if and only if no $s \in S$ is of order 2.

Proof.

We have a map

$$G \to \mathrm{Isom}(\Gamma(S, G)) \quad g \mapsto L_g$$

where $L_g \in \mathrm{Isom}(\Gamma(S, G))$ extends $L_g : G \to G$, $L_g(x) = gx$ to edges. □

Cayley Graphs

Definition

The restriction of d_S to $G \times G$ is called the word metric associated to S.

Exercises

- $|g|_S := d_S(1, g)$ *is the minimum length of a word w in S such that $g =_G w$.*
- $d_S(g, h)$ *is the minimum length of a word w in S such that $gw =_G h$.*

Proposition

If $G = \langle S \rangle = \langle \bar{S} \rangle$ then d_S and $d_{\bar{S}}$ are bi-Lipschitz equivalent. That is, there exists $L > 0$ such that

$$\frac{1}{L} d_S(g, h) \leq d_{\bar{S}}(g, h) \leq L d_S(g, h)$$

for every $g, h \in G$.

Cayley Graphs. Actions on simplicial trees

A simplicial tree is a connected graph with no monogons, digons or cycles.

Theorem

$\hat{\Gamma}(S, G)$ is a simplicial tree on which G acts freely $\iff$ $S = X \sqcup X^{-1}$, $G = F(X)$.

Proof.

Oriented paths in $\Gamma(S, G)$ without spikes correspond to pairs (g, w), w a reduced word in S.
$(\Leftarrow)$: A cycle would correspond to a reduced word $w = 1$ in $F(X)$.
$(\Rightarrow)$: G acts freely $\implies \forall s \in S$, $|\{s, s^{-1}\}| = 2$. For every such pair, pick one and together let these form X. X generates G and so there exists an onto homomorphism $\varphi : F(X) \to G$. Suppose $w \in F(X)$, $w \in \ker \varphi$. Since w is reduced as a word in X, it is also reduced as a word in S. So if $w \neq w_\emptyset$ then w gives a cycle in $\hat{\Gamma}(S, G)$. So $\ker \varphi = \{w_\emptyset\}$. □

Actions on simplicial trees

General theorem: G is free if and only if G acts freely by isometries on a simplicial tree T. The $(\Rightarrow)$ direction is given by the previous theorem.

For the $(\Leftarrow)$ direction, we use the following lemma.

Lemma

There exists $X \subseteq T$, X a tree, such that X contains exactly one vertex from each orbit.

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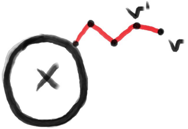
There exists $X \subseteq T$, X a tree, such that X contains exactly one vertex from each orbit.

Proof: Take X maximal such that X intersects each orbit $G \cdot v$ in at most one point (X exists by Zorn's lemma). Assume there exists some v such that $Gv \cap X = \emptyset$. Take v at minimal distance from X. If $d(v, X) = 1$, then we can add it to X - contradiction. So assume $d(v, X) \geq 2$.



By minimality, $gv' \in X$ for some $g \in G$. Therefore $d(gv, X) = 1$ and so we can add gv to X - contradiction. $\square$

Actions on simplicial trees

Lemma

There exists $X \subseteq T$, X a tree, such that X contains exactly one vertex from each orbit.

Theorem

G is free if and only if G acts freely by isometries on a simplicial tree T.

Actions on simplicial trees

Proof.

$(\Leftarrow)$: A 'tiling' of $V(T)$:

If $gX \cap X \neq \emptyset$ then there exists $v \in X$ such that $gv = v$ and so $g = 1$ by the freeness of the action. Hence if $g_1 \neq g_2$ then $g_1 X \cap g_2 X = \emptyset$.

Choose an orientation E^+ on the edges of T that is G-invariant. Let

$$S = \{g \in G : \exists e \in E^+, o(e) \in X, t(e) \in g(X)\}$$

We will prove that $G = F(S)$.

$\{g_1, g_2\}$ is an edge of $\hat{\Gamma}(S \cup S^{-1}, G)$ if and only if there exists an edge of T with one endpoint in $g_1 X$ and the other in $g_2 X$.

$\hat{\Gamma}(S \cup S^{-1}, G)$ is connected because T is. It is simplicial because it is a Cayley graph. And if $\hat{\Gamma}(S \cup S^{-1}, G)$ contains a cycle then so does T. So $G = F(S)$. □

Actions on simplicial trees

Theorem

G is free if and only if G acts freely by isometries on a simplicial tree T.

Corollary

Subgroups of free groups are free.

In order to study groups having actions on simplicial trees that are not free, we need the notion of amalgam.

Amalgams

Let A, B be groups with two isomorphic subgroups: i.e. there exist injective homomorphisms $\alpha : H \to A$, $\beta : H \to B$.

The amalgam of A and B over H is the "largest" group containing copies of A and B identified along H such that no other relation is imposed and such that it is generated by the copies of A and B.

We will define the amalgam by its universal property.

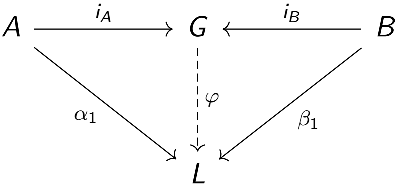
Amalgams

Notation: $\alpha(h) = h \in A$; $\beta(h) = \bar{h} \in B$.

Definition

G is the amalgamated product of A and B over H (written $G = A *_H B$) if

- there exist homomorphisms $i_A : A \to G$, $i_B : B \to G$ with $i_A(h) = i_B(\bar{h})$ for all $h \in H$;
- $\forall$ group L and $\forall$ homomorphisms $\alpha_1 : A \to L$, $\beta_1 : B \to L$ satisfying $\alpha_1(h) = \beta_1(\bar{h})$ for all $h \in H$, there exists a unique homomorphism $\varphi : G \to L$ such that $\alpha_1 = \varphi \circ i_A$ and $\beta_1 = \varphi \circ i_B$:



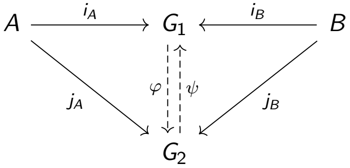
Amalgams

Remarks

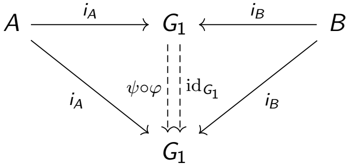
1. *The construction depends on the homomorphisms $\alpha : H \hookrightarrow A$, $\beta : H \hookrightarrow B$ but the notation is simplified.*
2. *It is not clear from the definition whether i_A and i_B are injective. However, this turns out to be the case.*

Uniqueness of the amalgam

Uniqueness of the amalgam: Suppose G_1 and G_2 are both amalgams of A, B over H. Then we have a commutative diagram



This implies that $\mathrm{id}_{G_1} : G_1 \to G_1$ and $\psi \circ \varphi : G_1 \to G_1$ both make the following diagram commute



And so $\psi \circ \varphi = \mathrm{id}_{G_1}$ by uniqueness of the induced homomorphism. Similarly $\varphi \circ \psi = \mathrm{id}_{G_2}$.

Existence of the amalgam

Existence of the amalgam:

Let $A = \langle S_1 | R_1 \rangle$, $B = \langle S_2 | R_2 \rangle$. WLOG $S_1 \cap S_2 = \emptyset$. Then

$$A *_H B = \langle S_1 \cup S_2 | R_1 \cup R_2 \cup \{h = \bar{h} : h \in H\} \rangle$$

Proof: Check that it satisfies the universal property (exercise).

Remarks

- *A and B generate $A *_H B$.*
- *i_A and i_B are injective.*

When $H = \{1\}$, the amalgam does not depend on α, β and it is called the free product of A and B, denoted by $A * B$.

Example

$F_2 = \mathbb{Z} * \mathbb{Z}$ *since if* $\mathbb{Z} = \langle a | \ \rangle$, $\mathbb{Z} = \langle b | \ \rangle$, *then* $\mathbb{Z} * \mathbb{Z} = \langle a, b | \ \rangle = F_2$.

Amalgams

We would like to describe the elements of $A *_H B$ by words.

Simplified notation: we identify H with $\alpha(H)$ and $\beta(H)$, and we identify A with $i_A(A)$, B with $i_B(B)$.

Let A_1 be a set of right coset representatives of H in A, and similarly let B_1 be a set of right coset representatives of H in B, such that $1 \in A_1$, $1 \in B_1$.

Definition

A reduced word of the amalgam $A *_H B$ is a word of the form $(h, s_1, ..., s_n)$, $h \in H$, $s_i \in A_1 \cup B_1$, $s_i \neq 1$, s_i alternating from A_1 to B_1. We associate to this the element $hs_1...s_n$ of $A *_H B$. The length of the reduced word is n.

Theorem

Each $g \in G = A *_H B$ is represented by a unique reduced word.

Amalgams

Theorem

Each $g \in G = A *_H B$ is represented by a unique reduced word.

Proof: For all $g \in G$, we can write $g = a_1 b_1 ... a_m b_m$ for some $a_i \in A$, $b_i \in B$.

We claim that g can be represented by a reduced word $(h, s_1, ..., s_n)$.

$m = 1$: $g = a_1 b_1 = a_1 \bar{h} b' = \underbrace{a_1 h}_{\in A} b' = h' a' b'$, where $a' \in A_1$, $b' \in B_1$.

Inductive step: exercise.

Uniqueness: Let X be the set of all reduced words. We will define an action of G on X, i.e. a group homomorphism

$$G \to Symm(X) = Bij(X)$$

Amalgams

By the universal property, it suffices to define $\alpha_1 : A \to Symm(X)$, $\beta_1 : B \to Symm(X)$ such that $\alpha_1(h) = \beta_1(\bar{h})$.

Definition of α_1: Consider $a \in A$.

Case 1: $a = h_0 \in H$:

$$h_0 \cdot (h, s_1, ..., s_n) = (h_0 h, s_1, ..., s_n)$$

Case 2: $a \in A \setminus H$.

2.a: $s_1 \in B$. $\forall h \in H$, write $ah = h'a'$ where $a' \in A_1$, $a' \neq 1$.

$$a \cdot (h, s_1, ..., s_n) = (h', a', s_1, ..., s_n)$$

2.b: $s_1 \in A$, $s_2 \in B$. $\forall h \in H$, write $ahs_1 = h'a'$, $a' \in A_1$.

$$\begin{aligned} a \cdot (h, s_1, ..., s_n) &= (h', a', s_2, ..., s_n) \quad \text{if } a' \neq 1 \\ &= (h', s_2, ..., s_n) \quad \text{if } a' = 1 \end{aligned}$$

Amalgams

This defines a map $\sigma_a : X \to X$.

Exercise: Check that $\sigma_{a_1 a_2} = \sigma_{a_1} \circ \sigma_{a_2}$.

Therefore $\sigma_a \circ \sigma_{a^{-1}} = \mathrm{id}$ and so σ_a is a bijection.

So we have defined $\alpha_1 : A \to Symm(X), \alpha_1(a) = \sigma_a$.

Likewise, we can define $\beta_1 : B \to Symm(X)$.

We have that $\alpha_1(h) = \beta_1(h) = \sigma_h$, for every $h \in H$.

Therefore there exists a unique $\varphi : A *_H B \to Symm(X)$.

Exercise: $\forall g \in G$, if $g = hs_1 ... s_n$, a reduced word, then

$$\varphi(g)(1) = (h, s_1, ..., s_n).$$

Thus, the reduced word is unique. □