A note on compactness of $(L^q)^* \hookrightarrow (W_0^{1,p})^*$

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $1 \le p < \infty$ and*

$$1 \le q < \begin{cases} \frac{np}{n-p} \text{ if } p < n, \\ \infty \text{ if } p \ge n, \end{cases}$$

so that the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact. Then $(L^q(\Omega))^* = L^{q'}(\Omega)$ is compactly embedded into $(W_0^{1,p}(\Omega))^* = W^{-1,p'}(\Omega)$.

Proof. Let $f \in (L^q(\Omega))^* = L^{q'}(\Omega)$. Then for every $\varphi \in W_0^{1,p}(\Omega)$,

$$\Big| \int_\Omega f\varphi\,dx \Big| \le \|f\|_{L^{q'}} \|\varphi\|_{L^q} \le C\|f\|_{L^{q'}} \|\varphi\|_{W^{1,p}}.$$

Hence f defines a bounded linear functional on $W_0^{1,p}(\Omega)$ (i.e. an element of $(W_0^{1,p}(\Omega))^*$ $= W^{-1,p'}(\Omega)$) and

$$\|f\|_{W^{-1,p'}} \le C\|f\|_{L^{q'}}.$$

This means that $L^{q'}(\Omega)$ is continuously embedded into $W^{-1,p'}(\Omega)$.

Suppose next that $(f_m) \subset L^{q'}(\Omega)$ is bounded. We need to show that there is a subsequence (f_{m_k}) which converges strongly in $W^{-1,p'}(\Omega)$. First, as $q' > 1$ and by Banach-Alaoglu's theorem, we can extract a subsequence (f_{m_k}) which converges weak* in $L^{q'}(\Omega)$ to some $f \in L^{q'}(\Omega)$, that is

$$\int_\Omega f_{m_k}\psi\,dx = \int_\Omega f\psi\,dx \text{ for all } \psi \in L^q(\Omega). \tag{1}$$

We claim that $\|f_{m_k} - f\|_{W^{-1,p'}} \to 0$. Recall that

$$\|f_{m_k} - f\|_{W^{-1,p'}} = \sup_{\varphi \in W_0^{1,p}(\Omega), \|\varphi\|_{W^{1,p}} \le 1} \Big| \int_\Omega (f_{m_k} - f)\varphi\,dx \Big|.$$

Let

$$K = \text{ closure of } \Big\{\varphi \in W_0^{1,p}(\Omega), \|\varphi\|_{W^{1,p}} \le 1\Big\} \subset L^q(\Omega) \text{ in } L^q(\Omega).$$

By hypothesis, K is a compact subset of $L^q(\Omega)$, and clearly,

$$\|f_{m_k} - f\|_{W^{-1,p'}} \le \sup_K \Big| \int_\Omega (f_{m_k} - f)\varphi\,dx \Big|. \tag{2}$$

Fix some $\delta > 0$. By compactness of K, we can find a finite number of balls $B_\delta(\psi_1), \dots, B_\delta(\psi_N) \subset L^q(\Omega)$ which cover K. By (1), there is some k_0 such that

$$\Big| \int_\Omega (f_{m_k} - f)\psi_i\,dx \Big| \le \delta \text{ for all } i = 1, \dots, N, k \ge k_0.$$

Now, if $\varphi \in K$, then $\varphi \in B_\delta(\psi_i)$ for some i and so, for $k \geq k_0$,

$$\begin{aligned}\Big|\int_\Omega (f_{m_k} - f)\varphi\,dx\Big| &\leq \Big|\int_\Omega (f_{m_k} - f)\psi_i\,dx\Big| + \int_\Omega |f_{m_k} - f||\varphi - \psi_i|\,dx \\ &\leq \delta + \|f_{m_k} - f\|_{L^{q'}}\|\varphi - \psi_i\|_{L^q} \\ &\leq \delta + 2\delta \sup_m \|f_m\|_{L^{q'}}.\end{aligned}$$

Returning to (2), we obtain

$$\|f_{m_k} - f\|_{W^{-1,p'}} \leq \delta + 2\delta \sup_m \|f_m\|_{L^{q'}}.$$

Since $\delta > 0$ is arbitrary and $\sup_m \|f_m\|_{L^{q'}} < \infty$, this implies that $\|f_{m_k} - f\|_{W^{-1,p'}} \to 0$.
□