Geometric Group Theory

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Amalgams

Theorem

Each $g \in G = A *_H B$ is represented by a unique reduced word.

Corollary

i_A and i_B are injective. Hence A, B can be seen as subgroups of $A *_H B$.

Corollary

If $(g_1, ..., g_n)$, $n \geq 2$, is such that $g_i \in A \cup B$, $g_i \notin H$, $\forall i \geq 2$, and g_i alternate between A and B, then $g_1 ... g_n \neq 1$ in $A *_H B$.

Proof.

Use induction to show that it can be represented by a reduced word of length $n-1$ if $g_1 \in H$ or of length n if $g_1 \notin H$. □

Amalgams

Theorem

Each $g \in G = A *_H B$ is represented by a unique reduced word.

Corollary

In G, $A \cap B = H$.

Definition

The reduced word $(h, s_1, ..., s_n)$ and the reduced element $hs_1 ... s_n \in A *_H B$ are cyclically reduced if $n \geq 2$ and $s_1 s_n$ is reduced.

Proposition

- *Every $g \in A *_H B$ is conjugate either to a cyclically reduced element or to some $a \in A$ or to some $b \in B$.*
- *Every cyclically reduced element has infinite order.*

Amalgams

Proposition

1. *Every $g \in A *_H B$ is conjugate either to a cyclically reduced element or to some $a \in A$ or to some $b \in B$.*
2. *Every cyclically reduced word has infinite order.*

Proof: (1) : If $g = hs_1 ... s_n$ is not cyclically reduced, i.e. s_1, s_n are both in A or both in B, then $s_n g s_n^{-1}$ is represented by a word of length $n - 1$.
Repeat until we have a cyclically reduced word or a word of length 1.

(2) : If g is cyclically reduced of length n then g^k has length kn, so $g^k \neq 1$. □

Corollary

Given any finite subgroup $F \leq A *_H B$, F must be contained in a conjugate gAg^{-1} or gBg^{-1}.

Proof: exercise.

The unique root property

Proposition

Every $u \in F(X)$ is conjugate to a cyclically reduced word.

Corollary (unique root property)

If $g, h \in F(X)$ are such that $g^k = h^k$ for some k then $g = h$.

Question: Find G torsion-free group s.t. $\exists g \neq h$ with $g^k = h^k$ for some k.
Take $G = \langle g, h \mid g^k = h^k \rangle$. It is an amalgamated product $G = A *_H B$, where $A = \langle g \rangle$, $B = \langle h \rangle$, and $H = \mathbb{Z} \simeq \langle g^k \rangle \simeq \langle h^k \rangle$.
Exercise: If every pair of distinct elements have an equal power then $G = \mathrm{Tor}\, G$. NB This does not mean that G is finite. See for instance https://en.wikipedia.org/wiki/Burnside_problem
Example due to Olshanskii: There exist finitely generated, non-cyclic, torsion-free groups G where any two elements have equal powers, i.e., for any g, h there exist m, n such that $g^m = h^n$.

Amalgams and actions on trees

Definition

- Suppose G is a group acting on a graph X. We say that G acts on X without inversions if for every $g \in G$ and $[v, w] \in E(X)$ we have that $g([v, w]) \neq [w, v]$.
- A free action of G on X is an action that is free on the vertices and without inversions.

Suppose G is a group acting on a tree T.

A subtree $S \subseteq T$ is a fundamental domain if it intersects the orbit $G \cdot v$ of every vertex v of T, and it intersects the orbit of every edge exactly once.

Theorem

$G = A *_H B$ acts on a tree T with fundamental domain an edge $[P, Q]$ such that $\mathrm{Stab}(P) = A$, $\mathrm{Stab}(Q) = B$, $\mathrm{Stab}([P, Q]) = H$.

Amalgams and actions on trees

Theorem

$G = A *_H B$ acts on a tree T with fundamental domain an edge $[P, Q]$ such that $\mathrm{Stab}(P) = A$, $\mathrm{Stab}(Q) = B$, $\mathrm{Stab}([P, Q]) = H$.

Proof:

Let $V(T) = G/A \sqcup G/B$.

Edges are (gA, gB), i.e. we join two left cosets of A and B if they have a common representative g. Given an edge, what is the set of common representatives corresponding to it?

$$g_1A = gA, \quad g_1B = gB \iff g^{-1}g_1 \in A \cap B = H$$

So the set is exactly gH. We label the edge (gA, gB) by gH and the edge (gB, gA) by $g\bar{H}$. Clearly, G acts transitively on the (non-oriented) edges and there are two orbits of vertices.

Amalgams and actions on trees

Let T be the non-oriented graph.

T is connected: For each edge $\{gA, gB\}$, $g = hs_1 \dots s_n$, we will prove it is connected by an edge path to $\{A, B\}$, by induction on n. Moreover, the length of the edge path (including $\{A, B\}$ and $\{gA, gB\}$) is $n + 1$. The case $n = 0$ is obvious.

Induction: if $s_n \in A_1 \setminus \{1\}$ then

$$gA = \underbrace{hs_1 \dots s_{n-1}}_{g'} A$$

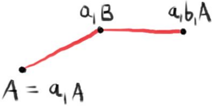
and $\{gA, gB\}$ shares an endpoint with $\{g'A, g'B\}$.
Similarly, if $s_n \in B_1 \setminus \{1\}$ then $gB = hs_1 \dots s_{n-1} B$ and $\{gA, gB\}$ has a common endpoint with $\{g'A, g'B\}$.

Amalgams and actions on trees

T is a tree: A path without spikes in T of origin A and even length $2n$ has vertices of the form:

$$A = a_1 A, a_1 B, a_1 b_1 A, ..., a_1 b_1 ... a_n b_n A$$

where $a_i \not\in H$ and $b_i \not\in H$.



An easy induction on n shows that the reduced form of $a_1 b_1 ... a_n b_n$ is $ha_1' b_1' ... a_n' b_n'$: for $n = 1$ we have

$$a_1 b_1 = a_1 \underbrace{hb_1'}_{b_1' \neq 1 \text{ as } b_1 \not\in H} = h' a_1' b_1' \quad \text{where} \quad a_1', b_1' \neq 1$$

Amalgams and actions on trees

Likewise,

$$a_1 b_1 a_2 b_2 ... a_{n+1} b_{n+1} = a_1 b_1 h a_2' b_2' ... a_{n+1}' b_{n+1}' = h' a_1' b_1' ... a_{n+1}' b_{n+1}'$$

In particular we cannot have $a_1 b_1 ... a_n b_n A = A$ otherwise

$$\underbrace{h a_1' b_1' ... a_n' b_n'}_{\text{length } 2n} = \underbrace{h' a'}_{\text{length } 0 \text{ or } 1}$$

So there is no cycle through A and so there is no cycle in T (every cycle must contain one vertex in G/A and so can be G-translated to a cycle through A). □

Amalgams and actions on trees

Theorem

$G = A *_H B$ acts on a tree T with fundamental domain an edge $[P, Q]$ such that $\mathrm{Stab}(P) = A$, $\mathrm{Stab}(Q) = B$, $\mathrm{Stab}([P, Q]) = H$.

Corollary

If $F \leq A *_H B$ is such that $F \cap gAg^{-1} = \{1\}$ and $F \cap gBg^{-1} = \{1\}$ for every $g \in G$, then F is free.

Proof.

F acts on the tree T without vertex or edge stabilisers and so it acts freely on the tree T. □

Proposition

The kernel of the map $\varphi : A * B \to A \times B$ is free.