Infinite groups: Sheet 2

October 10, 2024

Exercise 1. 1. Prove that if S and $\bar{S}$ are two finite generating sets of G, then the word metrics dist_S and $\mathrm{dist}_{\bar{S}}$ on G are bi-Lipschitz equivalent, i.e. there exists $L > 0$ such that

$$\frac{1}{L}\mathrm{dist}_S(g,g') \leqslant \mathrm{dist}_{\bar{S}}(g,g') \leqslant L\mathrm{dist}_S(g,g')\,, \forall g, g' \in G\,. \tag{1}$$

2. Prove that an isomorphism between two finitely generated groups is a bi-Lipschitz map when the two groups are endowed with word metrics.

Solution.

(1) It suffices to prove the inequality for $g' = e$, by left-invariance of the word metrics. Take L to be the maximum of $|\bar{s}|_S$, where $\bar{s} \in \bar{S}$ and of $|s|_{\bar{S}}$, where $s \in S$. Then every element in G that can be written as a word in S of length n can be written as a word of length at most Ln in $\bar{S}$; likewise every element in G that can be written as a word in $\bar{S}$ of length m can be written as a word of length at most Lm in S.

(2) An isomorphism $\varphi : G \to G'$ is even an isometry if we consider word metrics with respect to a finite generating set S for G and $\varphi(S)$ for G'. For different choices of generating sets it can be shown to be bi-Lipschitz, with an argument similar to that in **(1)**.

Exercise 2. Consider the *integer Heisenberg group*

$$H_{2n+1}(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & \dots & \dots & x_n & z \\ 0 & 1 & 0 & \dots & \dots & 0 & y_n \\ 0 & 0 & 1 & \dots & \dots & 0 & y_{n-1} \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 & y_2 \\ 0 & 0 & \dots & \dots & 0 & 1 & y_1 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \; ; \; x_1, \dots, x_n, y_1, \dots, y_n, z \in \mathbb{Z} \right\}.$$

Prove that $H_{2n+1}(\mathbb{Z})$ is nilpotent of class 2.

Solution. The multiplication of two matrices as above means the addition of the respective x coordinates and y coordinates and, in the upper right corner $z + z' + \sum_i x_i y'_{n-i}$. This immediately implies that a commutator has the coordinates x and y zero, hence $C^2 H_{2n+1}(\mathbb{Z})$ is composed of such matrices. The above description of the multiplication also implies that $C^2 H_{2n+1}(\mathbb{Z})$ is in the centre of $H_{2n+1}(\mathbb{Z})$, hence $C^3 H_{2n+1}(\mathbb{Z}) = \{1\}$.

Exercise 3. The goal of this exercise is to prove that, given an arbitrary field $\mathbb{K}$, the group $\mathcal{U}_n(\mathbb{K})$ is nilpotent of class $n-1$.

Let $\mathcal{U}_{n,k}(\mathbb{K})$ be the subset of $\mathcal{U}_n(\mathbb{K})$ formed by matrices (a_{ij}) such that $a_{ij} = \delta_{ij}$ for $j < i + k$. Note that $\mathcal{U}_{n,1}(\mathbb{K}) = \mathcal{U}_n(\mathbb{K})$.

1. Prove that for every $k \geqslant 1$ the map

$$\begin{array}{rcc} \varphi_k : \mathcal{U}_{n,k}(\mathbb{K}) & \to & \left(\mathbb{K}^{n-k}\,,\,+\right) \\ A = (a_{i,j}) & \mapsto & (a_{1,k+1}, a_{2,k+2}, \ldots, a_{n-k,n}) \end{array}$$

   is a homomorphism. Deduce that $(\mathcal{U}_{n,k}(\mathbb{K}))' \subset \mathcal{U}_{n,k+1}(\mathbb{K})$ and that $\mathcal{U}_{n,k+1}(\mathbb{K}) \lhd \mathcal{U}_{n,k}(\mathbb{K})$ for every $k \geqslant 1$.

2. Let E_{ij} be the matrix with all entries 0 except the (i,j)–entry, which is equal to 1. Consider the triangular matrix $T_{ij}(a) = I + aE_{ij}$.

   Deduce from (1), using induction, that $\mathcal{U}_{n,k}$ is generated by the set

$$\{T_{ij}(a) \mid j \geqslant i + k, a \in \mathbb{R}\}\,.$$

3. Prove that for every three distinct numbers i, j, k in $\{1, 2, \ldots, n\}$

$$[T_{ij}(a), T_{jk}(b)] = T_{ik}(ab)\,,\ [T_{ij}(a), T_{ki}(b)] = T_{kj}(-ab)\,,$$

   and that for all quadruples of distinct numbers i, j, k, ℓ,

$$[T_{ij}(a), T_{k\ell}(b)] = I\,.$$

4. Prove that $C^k \mathcal{U}_n(\mathbb{K}) \leqslant \mathcal{U}_{n,k}(\mathbb{K})$ for every $k \geqslant 0$. Deduce that $\mathcal{U}_n(\mathbb{K})$ is nilpotent.

Solution. All these are straightforward calculations with matrices.

Exercise 4. Which of the permutation groups S_n, for $n \geq 2$, are nilpotent? Which of these groups are solvable?

Solution. The group of even permutations A_n is simple for $n \geq 5$, so $A_n = (A_n)' = C^2 A_n$, as the latter two are normal (even characteristic) non-trivial subgroups. Therefore A_n is neither nilpotent nor solvable, hence S_n is neither nilpotent nor solvable for $n \geq 5$.

The group S_2 is abelian.

The group $S_3 \simeq D_6$, the group of isometries of the equilateral triangle, has $S_3' \simeq C_3$, so it is solvable, but $[S_3, C_3] = C_3$, so S_3 is not nilpotent.

For S_4, $S_3 \leq S_4$, therefore S_4 is not nilpotent. The derived subgroup S_4' is contained in A_4.

The group A_4 contains the normal subgroup

$$V_4 = \{id, (12)(34), (13)(24).(14)(23)\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

If we take $C_3 = \langle (123) \rangle$, the subgroup $V_4 C_3 \simeq V_4 \rtimes C_3$ has order $12 = |A_4|$, therefore it is equal to A_4. It follows that A_4 is solvable, hence so is S_4', hence S_4 is solvable as well.

Exercise 5. Let D_∞ be the infinite dihedral group. Recall that this group can be realized as the group of isometries of $\mathbb{Z}$, generated by the symmetry $s : \mathbb{R} \to \mathbb{R}, s(x) = -x$, and the translation $t : \mathbb{R} \to \mathbb{R}, t(x) = x + 1$, and as noted before $D_\infty = \langle t \rangle \rtimes \langle s \rangle$.

1. Give an example of two elements a, b of finite order in D_∞ such that their product ab is of infinite order.

2. Find $\operatorname{Tor} D_\infty$.

3. Is D_∞ a nilpotent group ? Is D_∞ polycyclic ?

4. Are any of the finite dihedral groups D_{2n} nilpotent?

Solution.

(1) For every $k \in \mathbb{Z}$, the isometry st^k is the symmetry with respect to $-\frac{k}{2}$. Examples are $a = s$ and $b = st^k$.

(2) We have the splitting into left cosets $D_\infty = \langle t \rangle \sqcup s\langle t \rangle$. The set $\operatorname{Tor} D_\infty$ equals the coset $s\langle t \rangle$.

(3) As $\operatorname{Tor} D_\infty$ is not a subgroup, D_∞ is not nilpotent. It is polycyclic, since $D_\infty \simeq \mathbb{Z} \rtimes \mathbb{Z}_2$.

(4) $C^2 D_{2n} = \langle t^2 \rangle$. Inductively, $C^k D_{2n} = \langle t^{2^k} \rangle$. Therefore, the group is nilpotent if and only if $n = 2^m$ for some positive integer m.

Exercise 6. Let $\mathcal{T}_n(\mathbb{K})$ be the group of invertible upper-triangular $n \times n$ matrices with entries in a field $\mathbb{K}$.

1. Prove that $\mathcal{T}_n(\mathbb{K})$ is a semidirect product of its nilpotent subgroup $\mathcal{U}_n(\mathbb{K})$ introduced in Exercise 3, and the subgroup of diagonal matrices.

2. Prove that, if $\mathbb{K}$ has zero characteristic, the subgroup of $\mathcal{T}_n(\mathbb{K})$ generated by $I + E_{12}$ and by the diagonal matrix with $(-1, 1, \ldots, 1)$ on the diagonal is isomorphic to the infinite dihedral group D_∞. Deduce that $\mathcal{T}_n(\mathbb{K})$ is not nilpotent.

Solution. **1.** The two subgroups intersect in $\{I\}$, $\mathcal{U}_n(\mathbb{K})$ is a normal subgroup, and the product between it and the subgroup of diagonal matrices is $\mathcal{T}_n(\mathbb{K})$.

2. Let H be this subgroup, $t = I + E_{12}$ and s the diagonal matrix with $(-1, 1, \ldots, 1)$ on the diagonal. We have that $sts = t^{-1}$ and we deduce that $H = \langle t \rangle \rtimes \langle s \rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}_2 \simeq D_\infty$. Since D_∞ is not nilpotent, $\mathcal{T}_n(\mathbb{K})$ is not nilpotent.