Infinite groups: Sheet 4

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Exercise 1. Let G be a polycyclic group. Suppose that every finite quotient group of G is nilpotent. Prove that G is nilpotent. [*Hint*: use 'Noetherian induction'. If $A \cong \mathbb{Z}^d$ is an abelian normal subgroup of G, think about the lower central series of G/A^p for a prime p.]

Solution. Arguing by 'Noetherian induction' we may suppose that G is not nilpotent, but G/K is nilpotent whenever $1 \neq K \lhd G$. Since G is not nilpotent, it is infinite, hence it has an abelian normal subgroup $A \cong \mathbb{Z}^d$ for some $d \geq 1$. Also G/A is nilpotent, so $C^c(G) \leq A$ for some c.

Let p be a prime. We have that $1 < A^p \lhd G$ so G/A^p is nilpotent. Since $A/A^p \cong \mathbb{F}_p^d$, G acts as a unipotent linear group on A/A^p and its image by this representation is nilpotent of class d. Therefore G satisfies

$$C^{c+d}(G) \leq [\ldots[[A, G], G,], \ldots, G] \leq A^p$$

(d brackets). But $\bigcap_p A^p = 1$ so $C^{c+d}(G) = 1$, the desired contradiction.

Exercise 2. Suppose that $G \leq \mathrm{GL}_n(\mathbb{K})$ is completely reducible and that $g^e = 1,\ \forall g \in G$. Prove that $|G| \leq e^{n^3}$. [*Hint*: first consider the irreducible case.]

Solution. Suppose first that G is *irreducible*. If $g \in G$ then each eigenvalue of g is an eth root of unity, so $tr(g)$ is the sum of n eth roots of unity. There are at most e such roots of unity, so there are at most e^n posible values for $tr(g)$. It follows by a result in lectures that $|G| \leq (e^n)^{n^2} = e^{n^3}$.

NB: In lectures and in Lecture Notes, the convention has been that the field $\mathbb{K}$ is always algebraically closed.

General case: G is isomorphic to a subgroup of $\pi_1(G) \times \cdots \times \pi_r(G)$ where each $\pi_i(G)$ is an irreducible group of degree n_i and $n_1 + \cdots + n_r = n$, and π_i is a homomorphism. Then $\pi_i(G)^e = 1$ so $|\pi_i(G)| \leq e^{n_i^3}$ for each i, and so

$$|G| \leq \prod e^{n_i^3} = e^{\sum n_i^3} \leq e^{n^3}$$

since $n^3 = (\sum n_i)^3 \geq \sum n_i^3$.

Exercise 3. Let G be a linear group. Prove that G is solvable if one of the following holds:

(i) every finitely generated subgroup of G is solvable;

(ii) G is finitely generated and every finite quotient group of G is solvable.

Solution. (i) Say G is linear of degree n. Then by the Zassenhaus Theorem every solvable subgroup of G has derived length at most $\beta = \beta(n)$. So $H^{(\beta)} = 1$ for every finitely generated subgroup H of G. It follows that $G^{(\beta)} = 1$, so G is solvable.

(ii) Now G is finitely generated. So G is residually finite. If $N \lhd G$ and G/N is linear of degree n over a finite field then G/N is finite, also solvable by hypothesis, so satisfies $(G/N)^{(\beta)} = 1$. Hence $G^{(\beta)} \leq N$, and as the subgroups N intersect in $\{1\}$ it follows that $G^{(\beta)} = 1$.

Exercise 4. Consider a semidirect product $\mathbb{Z}^n \rtimes_\varphi \mathbb{Z}$, defined by a homomorphism $\varphi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}^n) = GL(n, \mathbb{Z})$, hence by the matrix $\varphi(1) = M \in GL(n, \mathbb{Z})$. In what follows we use the notation $\mathbb{Z}^n \rtimes_M \mathbb{Z}$ instead of $\mathbb{Z}^n \rtimes_\varphi \mathbb{Z}$.

1. Prove, by induction on n, that if M has all eigenvalues equal to 1 then $\mathbb{Z}^n \rtimes_M \mathbb{Z}$ is nilpotent.

2. Deduce that, if M has all eigenvalues roots of unity, then $\mathbb{Z}^n \rtimes_M \mathbb{Z}$ is virtually nilpotent.

Solution. **(1)** For $n = 0$ there is nothing to prove; we assume, therefore, that the statement holds for $n - 1$. The matrix M has only eigenvalues equal to 1. Lemma 7.16 in the Lecture Notes then implies that there exists a finite ascending series

$$\{1\} = H_n \leqslant H_{n-1} \leqslant \ldots \leqslant H_1 \leqslant A = H_0 = \mathbb{Z}^n$$

such that $H_i \simeq \mathbb{Z}^{n-i}$, each quotient H_i/H_{i+1} is cyclic, the automorphism θ preserves each H_i and induces the identity automorphism on H_i/H_{i+1}. Thus, θ acts via the identity on H_{n-1}. In particular, the subgroup H_{n-1} is central in G; the automorphism θ projects to an automorphism $\bar{\theta} : \bar{A} \to \bar{A}$, $\bar{A} = A/H_{n-1}$. The automorphism $\bar{\theta}$ preserves the central series

$$\{1\} = \bar{H}_{n-1} \leqslant \ldots \leqslant \bar{H}_1 \cong \mathbb{Z}^{n-1},$$

(where $\bar{H}_i = H_i/H_{n-1}$) and induces trivial automorphism of each quotient

$$\bar{H}_i/\bar{H}_{i+1} \cong H_i/H_{i+1}.$$

By the induction hypothesis, the group

$$\bar{G} = \bar{A} \rtimes_{\bar{\theta}} \mathbb{Z} \cong G/H_{n-1},$$

is nilpotent. Since central coextensions of nilpotent groups are again nilpotent, we conclude that the group G is nilpotent as well.

(2) There exists a power M^k with all eigenvalues 1 and $\mathbb{Z}^n \rtimes_{M^k} \mathbb{Z} \simeq \mathbb{Z}^n \rtimes_M (k\mathbb{Z})$ is a finite index subgroup in $\mathbb{Z}^n \rtimes_M \mathbb{Z}$.

Exercise 5. Let G be a finitely generated nilpotent group and let $\varphi \in Aut(G)$. Prove that the polycyclic group $P = G \rtimes_\varphi \mathbb{Z}$ is

1. either virtually nilpotent;

2. or has exponential growth.

Solution. We note that replacing φ with a power will replace P with a finite-index subgroup, and, hence, will not affect the virtual nilpotency of P and its growth rate. The automorphism φ preserves the lower central series of G; let θ_i denote the restriction of φ to C^iG, $i \geqslant 1$. Then θ_i projects to an automorphism φ_i of the finitely generated abelian group $B_i := C^iG/C^{i+1}G$. The automorphism φ_i induces an automorphism ψ_i of $\operatorname{Tor} B_i$ and an automorphism $\overline{\varphi}_i$ of $B_i/\operatorname{Tor} B_i \simeq \mathbb{Z}^{m_i}$. Each choice of a basis for $B_i/\operatorname{Tor} B_i$ associates to the automorphism $\overline{\varphi}_i$ a matrix M_i in $GL(m_i, \mathbb{Z})$. All the conditions below are independent of the choice of a basis, therefore in what follows we assume that an arbitrary fixed basis is chosen in each $B_i/\operatorname{Tor} B_i$.

We have two cases to consider:

(1) All matrices M_i only have eigenvalues of absolute value 1; hence, all the eigenvalues are roots of unity (Lecture Notes). Then there exists N such that the matrices of the automorphisms $\bar{\varphi}_i^N$ have only eigenvalues equal to 1 and the induced automorphisms of finite abelian groups

$$\psi_i : \operatorname{Tor} B_i \to \operatorname{Tor} B_i$$

are all equal to the identity. Without loss of generality we may therefore assume that the matrices M_i of all the φ_i's have all eigenvalues equal to 1, and that all the ψ_i are the identity automorphisms.

Lemma 7.16 from Lecture Notes applied to each $\overline{\varphi}_i$ and to each $\psi_i = \mathrm{id}_{\operatorname{Tor} B_i}$, imply that the lower central series of G is a sub-series of a cyclic series

$$\{1\} = H_n \leqslant H_{n-1} \leqslant \ldots \leqslant H_1 \leqslant H_0 = G,$$

where each H_i/H_{i+1} is cyclic, φ preserves each H_i and induces the identity map on H_i/H_{i+1}. We denote by t the generator of the semidirect factor $\mathbb{Z}$ in the decomposition $P = G \rtimes \mathbb{Z}$. By the definition of the semidirect product, for every $g \in G$, $tgt^{-1} = \varphi(g)$. The fact that φ acts as the identity on each H_i/H_{i+1} implies that $t^k(hH_{i+1})t^{-k} = hH_{i+1}$ for every h in H_i; equivalently

$$[t^k, h] \in H_{i+1} \tag{1}$$

for every such h.

Since G contains the kernel $C^2P = [P, P]$ of the abelanization map $G \to \mathbb{Z}$, it follows that $C^2P \leqslant G$. We claim that for every $i \geqslant 0$, $[P, H_i] \subseteq H_{i+1}$. Indeed, consider an arbitrary commutator $[h, s]$, $h \in H_i, s \in P$. Since s has the form $s = gt^k$, with $g \in G$ and $k \in \mathbb{Z}$, we obtain:

$$[h, s] = [h, gt^k] = [h, g][g, [h, t^k]][h, t^k] \ .$$

According to (1), $[h, t^k] \in H_{i+1}$. Also, since the lower central series of G is a subseries of (H_i), there exists $r \geqslant 1$ such that $C^rG \geqslant H_i \geqslant H_{i+1} \geqslant C^{r+1}G$. Then, $h \in H_i \leqslant C^rG$ and

$$[h, g] \in C^{r+1}G \leqslant H_{i+1}.$$

Likewise, as $[h, t^k] \in H_{i+1} \leqslant C^rG$, the commutator

$$[g, [h, t^k]] \in C^{r+1}G \leqslant H_{i+1}.$$

By putting it all together, we conclude that $[h, s] \in H_{i+1}$ and, hence, $[P, H_i] \subseteq H_{i+1}$.

An easy induction now shows that $C^{i+2}P \leqslant H_i$ for every $i \geqslant 1$; in particular, $C^{n+2}P \leqslant H_n = \{1\}$. Therefore, P is virtually nilpotent.

(2) Assume that at least one matrix M_i has an eigenvalue with absolute value strictly greater than 1, in particular, $m_i \geqslant 2$. The group P contains the subgroup

$$P_i := C^iG \rtimes_{\theta_i} \mathbb{Z}.$$

Furthermore, the subgroup $C^{i+1}G$ is normal in P_i and

$$P_i/C^{i+1}G \simeq B_i \rtimes_{\varphi_i} \mathbb{Z},$$

where $B_i = C^iG/C^{i+1}G$. Lastly,

$$(B_i \rtimes_{\varphi_i} \mathbb{Z})/\mathrm{Tor}\, B_i \cong \mathbb{Z}^{m_i} \rtimes_{M_i} \mathbb{Z}.$$

According to Proposition 7.19 from the Lecture Notes, the group $\mathbb{Z}^{m_i} \rtimes_{M_i} \mathbb{Z}$ has exponential growth. Therefore, in view of Proposition 7.9, parts (a) and (c), the groups $B_i \rtimes_{\varphi_i} \mathbb{Z}$, $P_i/C^{i+1}G$, P_i, and, hence, P, all have exponential growth. Thus, in the case (2), S has exponential growth.

Exercise 6. Let G be a finitely generated group G of sub-exponential growth.

The goal of this exercise is to prove that for all $\beta_1, \dots, \beta_m, g \in G$, the set of conjugates

$$\{g^k \beta_i g^{-k} \mid k \in \mathbb{Z}, i = 1, \dots, m\}$$

generates a finitely generated subgroup $N \leqslant G$.

1. Prove that the statement for $m = 1$ implies the statement for every integer $m \geq 1$.

2. In what follows we therefore assume $m = 1$, we set $\alpha := \beta_1$ and let α_k denote $g^k \alpha g^{-k}$ for $k \in \mathbb{Z}$. The goal is to prove that finitely many elements in the set $\{\alpha_k \mid k \in \mathbb{Z}\}$ generate the subgroup N.

   Verify that

$$g\alpha^{s_0} g\alpha^{s_1} \cdots g\alpha^{s_m} = \alpha_1^{s_0} \alpha_2^{s_1} \cdots \alpha_{m+1}^{s_m} g^{m+1}.$$

3. Prove that if for every integer $m \geq 1$ the map

$$\mu = \mu_m : \prod_{i=0}^{m} \mathbb{Z}_2 \to G$$

$$\mu : (s_i) \mapsto g\alpha^{s_0} g\alpha^{s_1} \cdots g\alpha^{s_m} .$$

is injective then G must have exponential growth.

4. Deduce from the fact that $\mu_m, m \geq 1$, cannot be all injective the fact that N is finitely generated.

Solution. (1) It suffices to prove the lemma for $m = 1$, since N is generated by the subgroups

$$N_i = \langle g^k \beta_i g^{-k} \mid k \in \mathbb{Z} \rangle, \quad i = 1, \ldots, m.$$

(2) Easy calculation.

(3) If for every $m \in \mathbb{N}$ the map μ is injective then for each sequence (s_i) we have 2^{m+1} products as above, and if $g, g\alpha$ are in the set of generators of G, all these products are in $B_G(1, m+1)$. This contradicts the hypothesis that G has sub-exponential growth.

(4) We found that there exists some m and two distinct sequences $(s_i), (t_i)$ in $\prod_{i=0}^{m} \mathbb{Z}_2$ such that

$$g\alpha^{s_0} g\alpha^{s_1} \cdots g\alpha^{s_m} = g\alpha^{t_0} g\alpha^{t_1} \cdots g\alpha^{t_m} . \tag{2}$$

Assume that m is minimal with this property. This, in particular, implies that $s_0 \neq t_0$ and $s_m \neq t_m$. In view of the exercise, the equality (2) becomes

$$\alpha_1^{s_0} \alpha_2^{s_1} \cdots \alpha_{m+1}^{s_m} = \alpha_1^{t_0} \alpha_2^{t_1} \cdots \alpha_{m+1}^{t_m} .$$

Since $s_m \neq t_m$ and $s_m, t_m \in \{0, 1\}$, it follows that $s_m - t_m = \pm 1$. Then

$$\alpha_{m+1}^{\pm 1} = \alpha_m^{-s_{m-1}} \cdots \alpha_2^{-s_1} \alpha_1^{t_0 - s_0} \alpha_2^{t_1} \cdots \alpha_m^{t_{m-1}} . \tag{3}$$

If in (3) we conjugate by g, we obtain that

$$\alpha_{m+2}^{\pm 1} = \alpha_{m+1}^{-s_{m-1}} \cdots \alpha_3^{-s_1} \alpha_2^{t_0 - s_0} \alpha_3^{t_1} \cdots \alpha_{m+1}^{t_{m-1}} .$$

This and (3) imply that α_{m+2} is a product of powers of $\alpha_1, \ldots, \alpha_m$. Then, by induction, every α_n with $n \in \mathbb{N}$ is a product of powers of $\alpha_1, \ldots, \alpha_m$, and the same is true for α_n with $n \in \mathbb{Z}$ by replacing g with g^{-1}. Therefore, every generator α_n of N belongs to the subgroup of N generated by the elements $\alpha_1, \ldots, \alpha_m$ and the elements $\alpha_1, \ldots, \alpha_m$ generate N.