Geometric Group Theory

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A quotation

Leo Moser: “A mathematician named Klein

Thought the Möbius band was divine.

Said he: "If you glue

The edges of two,

You'll get a weird bottle like mine.”

Indeed, by glueing two Möbius bands one obtains a Klein bottle.

There is a second way of obtaining a Klein bottle, that relates to a construction that we will introduce today: the HNN extension.

Amalgams and actions on trees

Theorem

$G = A *_H B$ acts on a tree T with fundamental domain an edge $[P, Q]$ such that $\mathrm{Stab}(P) = A$, $\mathrm{Stab}(Q) = B$, $\mathrm{Stab}([P, Q]) = H$.

Corollary

If $F \leq A *_H B$ is such that $F \cap gAg^{-1} = \{1\}$ and $F \cap gBg^{-1} = \{1\}$ for every $g \in G$, then F is free.

Proposition

The kernel of the map $\varphi : A * B \to A \times B$ is free.

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Proposition

The kernel of the map $\varphi : A * B \to A \times B$ is free.

Proof.

$F = \ker \varphi$ intersects gAg^{-1} and gBg^{-1} trivially. $\square$

Corollary

If A, B finite, then $A * B$ is virtually free.

A group is virtually (*) if it has a finite index subgroup that is (*).

Amalgams and actions on trees

Theorem

Suppose $G \curvearrowright T$ with fundamental domain an edge $e = [P, Q]$. If $A = \mathrm{Stab}(P)$, $B = \mathrm{Stab}(Q)$, $H = \mathrm{Stab}(e)$ then $G = A *_H B$.

Proof

Since we have $\alpha_1 : A \to G$, $\beta_1 : B \to G$ agreeing on H, there exists some $\varphi : A *_H B \to G$, by the Universal Property of $A *_H B$.

Step 1: $G = \langle A, B \rangle$, that is, φ is onto.

For all $g \in G$, ge is joined to e by a unique path of length n (counting e and ge). We will prove that $g \in \langle A, B \rangle$ by induction on n. If $n = 1$ then $g \in H$.

Assume true for n, and let ge be joined to e by a path of length $n + 1$. Let $g'e$ be the previous edge on the path.

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Then either $gP = g'P$ or $gQ = g'Q$ and so $g^{-1}g' \in A \cup B$. Since $g' \in \langle A, B \rangle$ we are done.

Step 2: φ is injective.

Let $hs_1 \dots s_n \in \ker \varphi$. We can prove by induction on n that $hs_1 \dots s_n e$ can be joined to e by an edge path with no spikes of length $n+1$. Hence $hs_1 \dots s_n \neq 1$ in G. □

HNN extensions

Definition

Suppose we have $A \subseteq G$ and $\theta : A \to G$ an injective homomorphism. The HNN extension of G on A with respect to θ is

$$\begin{aligned} G*_A &:= \langle G, t | tat^{-1} = \theta(a), \forall a \in A\rangle \\ &= G * \langle t\rangle \ / \ \langle\langle \{tat^{-1}\theta(a)^{-1} : a \in A\}\rangle\rangle \end{aligned}$$

t is called the stable letter of the HNN extension.

The name comes from Graham Higman, Bernhard Neumann and Hanna Neumann.

A definition with a Universal Property can be formulated using

$$G*_A = \langle G, t | tat^{-1} = \theta(a), \forall a \in A\rangle.$$

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Remark

If $G = \langle S|R \rangle$ *then* $G*_A = \langle S \cup \{t\} | R \cup \{tat^{-1} = \theta(a) : a \in A\} \rangle$.

Examples

- *The Baumslag-Solitar groups* $BS(m, n) = \langle a, t | ta^m t^{-1} = a^n \rangle$, *where* $m, n \in \mathbb{Z}$.
- *When* $m = n = 1$, *we have* $\mathbb{Z}^2$ *(fundamental group of the torus.)*
- *When* $m = 1$, $n = -1$, *the fundamental group of the Klein bottle.*
- *When* $m = 1$ *(or* $n = 1$*) the group is solvable.*

Reduced words of HNN extensions

Suppose A_1 is a set of right coset representatives for A and A_2 is a set of right coset representatives for $\theta(A)$ such that $1 \in A_1 \cap A_2$.

A reduced word of $G*_A$ is a sequence $(g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, ..., t^{\epsilon_n}, g_n)$ such that

- $\epsilon = \pm 1$
- $g_0 \in G$
- $g_i \in A_1$ if $\epsilon_i = 1$, $g_i \in A_2$ if $\epsilon_i = -1$
- $g_i \neq 1$ if $\epsilon_{i+1} = -\epsilon_i$

A reduced element of $G*_A$ is an element of the form $g_0 t^{\epsilon_1} g_1 ... t^{\epsilon_n} g_n$.

Theorem

Each $g \in G*_A$ is represented by a unique reduced word.

Reduced words of HNN extensions

Theorem

Each $g \in G*_A$ is represented by a unique reduced word.

Proof:

Existence of a representation: We induct on the length of g as a reduced word in $G \cup \{t, t^{-1}\}$. The length 1 case is obvious.

Assume true for n. Length $n+1$ means either $g = ut^{\pm 1}$, $\mathrm{length}(u) \leq n$, or

$$g \in \{wth, wt^{-1}h\}$$

where $\mathrm{length}(w) \leq n-1$ and $h \in G$. If $g = ut^{\pm 1}$, apply induction. If

$$g = wth = wtah_1 = wtat^{-1}th_1 = w\theta(a)th_1$$

then $\mathrm{length}(w\theta(a)) \leq n$ so we can apply the inductive assumption. The case $g = wt^{-1}h$ is similar.

Reduced words of HNN extensions

Uniqueness of representation: Let X be the set of reduced words. $G*_A$ acts on it (i.e. there exists a group homomorphism $G*_A \to Bij(X)$) as follows:

$$\phi(g)(g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n) = (gg_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n)$$

and $\phi(t)(g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n)$ equals

$$\begin{cases} (\theta(g_0), t, 1, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n) & \text{if } g_0 \in A \text{ and } \epsilon_1 = 1 \\ (\theta(g_0)g_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g_n) & \text{if } g_0 \in A \text{ and } \epsilon_1 = -1 \\ (\theta(a), t, g_0', t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n) & \text{if } g_0 = ag_0' \text{ and } g_0' \in A_1 \setminus \{1\} \end{cases}$$

Exercise: Prove that $\phi(t)$ is a bijection. For instance, prove that $\phi(t^{-1})$ is its inverse.

We thus have a homomorphism $\phi : G * \langle t \rangle \to Bij(X)$.

Exercise: prove that $\phi(tat^{-1}) = \phi(\theta(a))$, $\forall a \in A$.

Hence ϕ defines $\bar{\phi} : G*_A \to Bij(X)$. And if $g = g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_n} g_n$ then $\phi(g)(1) = (g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n)$. □

HNN extensions

Theorem

Each $g \in G*_A$ is represented by a unique reduced word.

Corollary

The group G embeds into $G*_A$.

Corollary (Britton's lemma)

If $g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_n} g_n$ is such that $g_i \in G \setminus A$ when $(\epsilon_i, \epsilon_{i+1}) = (1, -1)$ and $g_i \in G \setminus \theta(A)$ when $(\epsilon_i, \epsilon_{i+1}) = (-1, 1)$ then it is non-trivial.

Graphs of groups

Definition

If $G = A *_H B$ or $G = A*_H$ then we say that G splits over H.

Definition

Let Y be an oriented graph such that the corresponding unoriented graph is connected and each of its edges appears with both orientations in Y.

A graph of groups is a pair (G, Y), where G is a map that assigns a group G_v to each vertex $v \in V(Y)$ and a group G_e to each edge $e \in E(Y)$ such that

1. $G_e = G_{\bar{e}}$
2. for all edges e, there exists an injective homomorphism $\alpha_e : G_e \to G_{t(e)}$

where $t(e)$ is the terminus of the edge $e = [o(e), t(e)]$.

Graphs of groups

Graphs of groups appear naturally when G acts on a graph X without inversions.

When this happens, we define the quotient graph $Y = X/G$ and the projection $p : X \to Y$ as follows:

- Vertices are orbits Gv, $v \in X$
- Gv, Gw are joined if there exists an edge $[v_1, w_1]$ such that $v_1 \in Gv$, $w_1 \in Gw$.

We define $p : X \to X/G$ by $p(v) = Gv$, $p(e) = \{Go(e), Gt(e)\}$.

In this case,

- $\forall v \in Y$, define $G_v = \mathrm{Stab}(\hat{v})$ where $\hat{v}$ is some element of $p^{-1}(v)$
- $\forall e \in Y$, define $G_e = \mathrm{Stab}(\hat{e})$ where $\hat{e}$ is some element of $p^{-1}(e)$

taking care that, whenever we can, $\hat{v}$ is an endpoint of $\hat{e}$ such that $G_e \subseteq G_v$.

For some edges, we might have to define α_e not as an inclusion, but as an inclusion composed with a conjugation.