Why study the history of mathematics?

Christopher Hollings

Mathematical Institute & The Queen's College
University of Oxford

Fridays@2, HT25, Week 6

Mathematical Institute

Oxford Mathematics

Some possible answers

Studying the history of mathematics

- humanises and contextualises mathematics
- makes mathematics more accessible (outreach)
- can promote diversity in mathematics
- promotes links between pure and applied mathematics
- promotes links with other disciplines
- can highlight different approaches to problem solving
- helps in understanding why we now do things in a certain way
- can aid in teaching/learning mathematics
- can spark new lines of mathematical research
- is interesting in its own right!

Some possible answers

Studying the history of mathematics

- humanises and contextualises mathematics
- makes mathematics more accessible (outreach)
- can promote diversity in mathematics
- promotes links between pure and applied mathematics
- promotes links with other disciplines
- can highlight different approaches to problem solving
- helps in understanding why we now do things in a certain way
- can aid in teaching/learning mathematics
- can spark new lines of mathematical research
- is interesting in its own right!

Euclidean mathematics

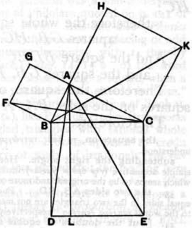
PROPOSITION 47.

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Let *ABC* be a right-angled triangle having the angle *BAC* right;

I say that the square on *BC* is equal to the squares on *BA*, *AC*.

For let there be described on *BC* the square *BDEC*, and on *BA*, *AC* the squares *GB*, *HC*; [I. 46] through *A* let *AL* be drawn parallel to either *BD* or *CE*, and let *AD*, *FC* be joined.



Then, since each of the angles *BAC*, *BAG* is right, it follows that with a straight line *BA*, and at the point *A* on it, the two straight lines *AC*, *AG* not lying on the same side make the adjacent angles equal to two right angles;

therefore *CA* is in a straight line with *AG*. [I. 14]

For the same reason

BA is also in a straight line with *AH*.

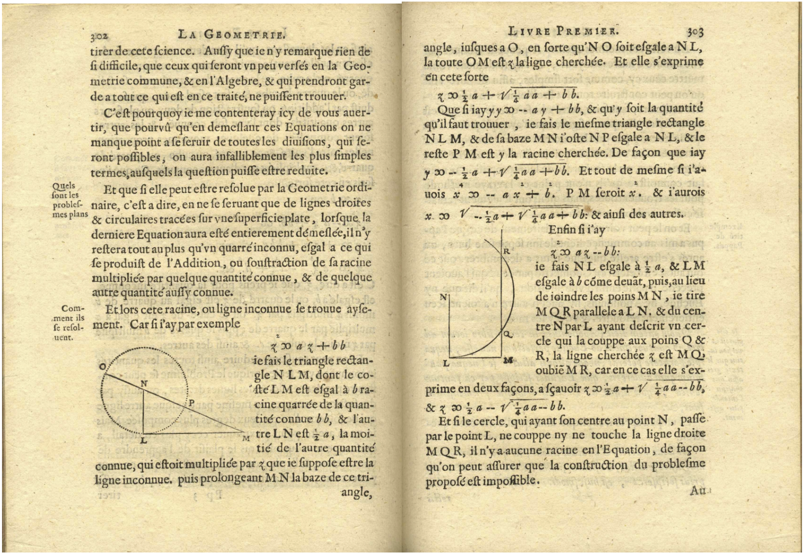
And, since the angle *DBC* is equal to the angle *FBA*: for each is right:

let the angle *ABC* be added to each;

therefore the whole angle *DBA* is equal to the whole angle *FBC*. [*C. N.* 2]

T. L. Heath, *The thirteen books of Euclid's Elements*, CUP, 1908

Cartesian mathematics



tirer de cete ſcience. Auſſy que ie n'y remarque rien de ſi difficile, que ceux qui ſeront vn peu verſés en la Geometrie commune, & en l'Algebre, & qui prendront garde a tout ce qui eſt en ce traité, ne puiſſent trouuer.

C'eſt pourquoy ie me contenteray icy de vous auertir, que pourvû qu'en demeſlant ces Equations on ne manque point a ſe ſeruir de toutes les diuiſions, qui ſeront poſſibles, on aura infalliblement les plus ſimples termes, auſquels la queſtion puiſſe eſtre reduite.

Quels ſont les probleſmes plans

Et que ſi elle peut eſtre reſolue par la Geometrie ordinaire, c'eſt a dire, en ne ſe ſeruant que de lignes droites & circulaires tracées ſur vne ſuperficie plate, lorſque la derniere Equation aura eſté entierement démeſlée, il n'y reſtera tout au plus qu'vn quarré inconnu, eſgal a ce qui ſe produiſt de l'Addition, ou ſouſtraction de ſa racine multipliée par quelque quantité connue, & de quelque autre quantité auſſy connue.

Comment ils ſe reſoluent.

Et lors cete racine, ou ligne inconnue ſe trouue ayſement. Car ſi i'ay par exemple

$$z^2 \infty az + bb$$

ie fais le triangle rectangle N L M, dont le coſté L M eſt eſgal à b racine quarrée de la quantité connue bb, & l'autre L N eſt $\frac{1}{2}a$, la moitié de l'autre quantité connue, qui eſtoit multipliée par z que ie ſuppoſe eſtre la ligne inconnue. puis prolongeant M N la baze de ce triangle,

angle, iuſques a O, en ſorte qu'N O ſoit eſgale a N L, la toute O M eſt z la ligne cherchée. Et elle s'exprime en cete ſorte

$$z \infty \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + bb}.$$

Que ſi iay $yy \infty -- ay + bb$, & qu'y ſoit la quantité qu'il faut trouuer, ie fais le meſme triangle rectangle N L M, & de ſa baze M N i'oſte N P eſgale a N L, & le reſte P M eſt y la racine cherchée. De façon que iay $y \infty -- \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$. Et tout de meſme ſi i'auois $x^4 \infty -- ax^2 + b^2$. P M ſeroit x^2. & i'aurois $x \infty \sqrt{-- \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}}$: & ainſi des autres.

Enfin ſi i'ay

$$z^2 \infty az -- bb:$$

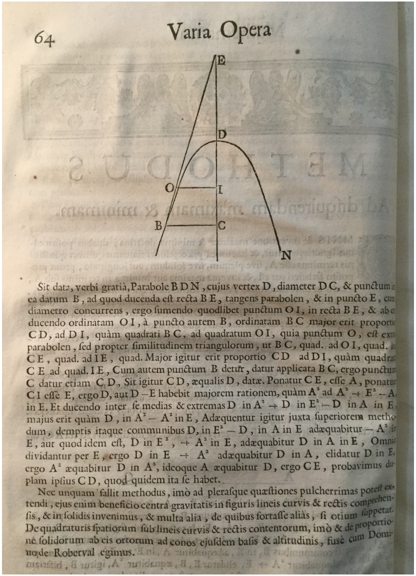
ie fais N L eſgale à $\frac{1}{2}a$, & L M eſgale à b cõme deuãt, puis, au lieu de ioindre les poins M N, ie tire M Q R parallele a L N. & du centre N par L ayant deſcrit vn cercle qui la couppe aux poins Q & R, la ligne cherchée z eſt M Q, oubiẽ M R, car en ce cas elle s'exprime en deux façons, a ſçauoir $z \infty \frac{1}{2}a + \sqrt{\frac{1}{4}aa -- bb}$, & $z \infty \frac{1}{2}a -- \sqrt{\frac{1}{4}aa -- bb}$.

Et ſi le cercle, qui ayant ſon centre au point N, paſſe par le point L, ne coupe ny ne touche la ligne droite M Q R, il n'y a aucune racine en l'Equation, de façon qu'on peut aſſurer que la conſtruction du probleſme propoſé eſt impoſſible.

Au

René Descartes, *La géométrie*, Leiden, 1637

Fermat's tangent method

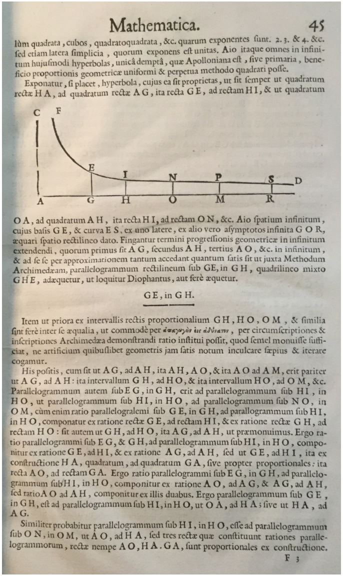


Sit data, verbi gratià, Parabole B D N, cujus vertex D, diameter D C, & punctum in ea datum B, ad quod ducenda eſt recta B E, tangens parabolen, & in puncto E, cum diametro concurrens, ergo ſumendo quodlibet punctum O I, in recta B E, & ab eo ducendo ordinatam O I, à puncto autem B, ordinatam B C major erit proportio C D, ad D I, quàm quadrati B C, ad quadratum O I, quia punctum O, eſt extra parabolen, ſed propter ſimilitudinem triangulorum, ut B C, quad. ad O I, quad. ita C E, quad. ad I E, quad. Major igitur erit proportio C D ad D I, quàm quadrati C E ad quad. I E, Cum autem punctum B detur, datur applicata B C, ergo punctum C datur etiam C D, Sit igitur C D, æqualis D, datæ. Ponatur C E, eſſe A, ponatur C I eſſe E, ergo D, aut D – E habebit majorem rationem, quàm A^2 ad $A^2 + E^2 - A$, in E. Et ducendo inter ſe medias & extremas D in A^2 + D in E^2 – D in A in E, majus erit quàm D, in A^2 – A^2 in E, Adæquentur igitur juxta ſuperiorem methodum, demptis itaque communibus D, in E^2 – D, in A in E adæquabitur – A^2 in E, aut quod idem eſt, D in E^2, + A^2 in E, adæquabitur D in A in E, Omnia dividantur per E, ergo D in E + A^2 adæquabitur D in A, elidatur D in E, ergo A^2 æquabitur D in A^2, ideoque A æquabitur D, ergo C E, probavimus duplam ipſius C D, quod quidem ita ſe habet.

Nec unquam fallit methodus, imò ad pleraſque quæſtiones pulcherrimas poteſt extendi, ejus enim beneficio centra gravitatis in figuris lineis curvis & rectis comprehenſis, & in ſolidis invenimus, & multa alia, de quibus fortaſſe aliàs, ſi otium ſuppetat. De quadraturis ſpatiorum ſub lineis curvis & rectis contentorum, imò & de proportione ſolidorum ab eis ortorum ad conos ejuſdem baſis & altitudinis, fusè cum Domino de Roberval egimus.

Worked out *c.* 1629, but only published posthumously in *Varia opera mathematica*, 1679

Fermat's quadrature of higher hyperbolas



Mathematica. 45

lùm quadrata, cubos, quadratoquadrata, &c. quarum exponentes ſunt. 2. 3. & 4. &c. ſed etiam latera ſimplicia, quorum exponens eſt unitas. Aio itaque omnes in infinitum hujuſmodi hyperbolas, unicâ demptâ, quæ Apolloniana eſt, ſive primaria, beneficio proportionis geometricæ uniformi & perpetua methodo quadrari poſſe.

Exponatur, ſi placet, hyperbola, cujus ea ſit proprietas, ut ſit ſemper ut quadratum rectæ H A, ad quadratum rectæ A G, ita recta G E, ad rectam H I, & ut quadratum O A, ad quadratum A H, ita recta H I, ad rectam O N, &c. Aio ſpatium infinitum, cujus baſis G E, & curva E S, ex uno latere, ex alio vero aſymptotos infinita G O R, æquari ſpatio rectilineo dato. Fingantur termini progreſſionis geometricæ in infinitum extendendi, quorum primus ſit A G, ſecundus A H, tertius A O, &c. in infinitum, & ad ſe ſe per approximationem tantum accedant quantum ſatis ſit ut juxta Methodum Archimedæam, parallelogrammum rectilineum ſub GE, in G H, quadrilineo mixto G H E, adæquetur, ut loquitur Diophantus, aut ferè æquetur.

G E, in G H.

Item ut priora ex intervallis rectis proportionalium G H, H O, O M, & ſimilia ſint ferè inter ſe æqualia, ut commodè per ἀπαγωγὴν εἰς ἀδύνατον, per circumſcriptiones & inſcriptiones Archimedæa demonſtrandi ratio inſtitui poſſit, quod ſemel monuiſſe ſufficiat, ne artificium quibuſlibet geometris jam ſatis notum inculcare ſæpius & iterare cogamur.

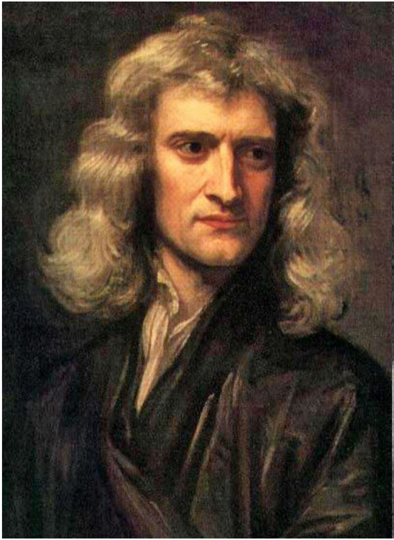
His poſitis, cum ſit ut A G, ad A H, ita A H, A O, & ita A O ad A M, erit pariter ut A G, ad A H: ita intervallum G H, ad H O, & ita intervallum H O, ad O M, &c. Parallelogrammum autem ſub E G, in G H, erit ad parallelogrammum ſub H I, in H O, ut parallelogrammum ſub H I, in H O, ad parallelogrammum ſub N O, in O M, cùm enim ratio parallelogralemi ſub G E, in G H, ad parallogrammum ſub H I, in H O, componatur ex ratione rectæ G E, ad rectam H I, & ex ratione rectæ G H, ad rectam H O: ſit autem ut G H, ad H O, ita A G, ad A H, ut præmonuimus. Ergo ratio parallelogrammi ſub E G, & G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione G E, ad H I, & ex ratione A G, ad A H, ſed ut G E, ad H I, ita ex conſtructione H A, quadratum, ad quadratum G A, ſive propter proportionales: ita recta A O, ad rectam G A. Ergo ratio parallelogrammi ſub E G, in G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione A O, ad A G, & A G, ad A H, ſed ratio A O ad A H, componitur ex illis duabus. Ergo parallelogrammum ſub G E, in G H, eſt ad parallelogrammum ſub H I, in H O, ut O A, ad H A; ſive ut H A, ad A G.

Similiter probabitur parallelogrammum ſub H I, in H O, eſſe ad parallelogrammum ſub O N, in O M, ut A O, ad H A, ſed tres rectæ quæ conſtituunt rationes parallelogrammorum, rectæ nempe A O, H A. G A, ſunt proportionales ex conſtructione.

F 3

Worked out in the early 1640s, but only published posthumously in *Varia opera mathematica*, 1679

The birth of calculus



Newton's method of fluxions

12. Ex. 5. As if the Equation $zz + axz - y^4 = 0$ were propos'd to expreſs the Relation between x and y, as alſo $\sqrt{ax-xx} = BD$, for determining a Curve, which therefore will be a Circle. The Equation $zz + axz - y^4 = 0$, as before, will give $2\dot{z}z + a\dot{x}z + ax\dot{z} - 4\dot{y}y^3 = 0$, for the Relation of the Celerities $\dot{x}$, $\dot{y}$, and $\dot{z}$. And therefore ſince it is $\dot{z} = \dot{x} \times BD$ or $= \dot{x}\sqrt{ax-xx}$, ſubſtitute this Value inſtead of it, and there will ariſe the Equation $\overline{2\dot{x}z + ax\dot{x}}\sqrt{ax-xx} + a\dot{x}z - 4\dot{y}y^3 = 0$, which determines the Relation of the Celerities $\dot{x}$ and $\dot{y}$.

DEMONSTRATION *of the Solution.*

13. The Moments of flowing Quantities, (that is, their indefinitely ſmall Parts, by the acceſſion of which, in indefinitely ſmall portions of Time, they are continually increaſed,) are as the Velocities of their Flowing or Increaſing.

14. Wherefore if the Moment of any one, as x, be repreſented by the Product of its Celerity $\dot{x}$ into an indefinitely ſmall Quantity o (that is, by $\dot{x}o$,) the Moments of the others v, y, z, will be repreſented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; becauſe $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$.

See the Analise des infiniment petites of the M. de L'Hospital.

15. Now ſince the Moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little acceſſions of the flowing Quantities x and y, by which thoſe Quantities are increaſed through the ſeveral indefinitely little intervals of Time; it follows, that thoſe Quantities x and y, after any indefinitely ſmall interval of Time, become $x + \dot{x}o$ and $y + \dot{y}o$. And therefore the Equation, which at all times indifferently expreſſes the Relation of the flowing Quantities, will as well expreſs the Relation between $x + \dot{x}o$ and $y + \dot{y}o$, as between x and y: So that $x + \dot{x}o$ and $y + \dot{y}o$ may be ſubſtituted in the ſame Equation for thoſe Quantities, inſtead of x and y.

16. Therefore let any Equation $x^3 - ax^2 + axy - y^3 = 0$ be given, and ſubſtitute $x + \dot{x}o$ for x, and $y + \dot{y}o$ for y, and there will ariſe

$$\left.\begin{array}{llll} x^3 & + 3\dot{x}ox^2 & + 3\dot{x}^2oox & + \dot{x}^3o^3 \\ - ax^2 & - 2a\dot{x}ox & - a\dot{x}^2oo & \\ + axy & + a\dot{x}oy & + a\dot{y}ox & + a\dot{x}\dot{y}oo \\ - y^3 & - 3\dot{y}oy^2 & - 3\dot{y}^2ooy & - \dot{y}^3o^3 \end{array}\right\} = 0.$$

17.

17. Now by Suppoſition $x^3 - ax^2 + axy - y^3 = 0$, which therefore being expunged, and the remaining Terms being divided by o, there will remain $3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3oo - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3oo = 0$. But whereas o is ſuppoſed to be infinitely little, that it may repreſent the Moments of Quantities; the Terms that are multiply'd by it will be nothing in reſpect of the reſt. Therefore I reject them, and there remains $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$, as above in Examp. 1.

18. Here we may obſerve, that the Terms that are not multiply'd by o will always vaniſh, as alſo thoſe Terms that are multiply'd by o of more than one Dimenſion. And that the reſt of the Terms being divided by o, will always acquire the form that they ought to have by the foregoing Rule: Which was the thing to be proved.

19. And this being now ſhewn, the other things included in the Rule will eaſily follow. As that in the propos'd Equation ſeveral flowing Quantities may be involved; and that the Terms may be multiply'd, not only by the Number of the Dimenſions of the flowing Quantities, but alſo by any other Arithmetical Progreſſions; ſo that in the Operation there may be the ſame difference of the Terms according to any of the flowing Quantities, and the Progreſſion be diſpos'd according to the ſame order of the Dimenſions of each of them. And theſe things being allow'd, what is taught beſides in Examp. 3, 4, and 5, will be plain enough of itſelf.

PROB. II.

An Equation being propoſed, including the Fluxions of Quantities, to find the Relations of thoſe Quantities to one another.

A PARTICULAR SOLUTION.

1. As this Problem is the Converſe of the foregoing, it muſt be ſolved by proceeding in a contrary manner. That is, the Terms multiply'd by $\dot{x}$ being diſpoſed according to the Dimenſions of x; they muſt be divided by $\frac{\dot{x}}{x}$, and then by the number of their Dimenſions, or perhaps by ſome other Arithmetical Progreſſion. Then the ſame work muſt be repeated with the Terms multiply'd by $\dot{v}$, $\dot{y}$,

E or

The method of fluxions and infinite series, London, 1736

Newton's method of fluxions

We begin with a fluent quantity x, assumed implicitly to depend upon some independent variable, and we seek the fluxion $\dot{x}$, i.e., the rate of change of x with respect to the independent variable

Newton employed the notion of a moment o, an 'indefinitely small Quantity'

When the independent variable changes by o, x changes by $\dot{x}o$

Newton's method of fluxions

We seek the fluxion of the equation $x^3 - ax^2 + axy - y^3 = 0$

Substitute $x + \dot{x}o$ for x and $y + \dot{y}o$ for y, and expand:

$$\left.\begin{array}{cccccccc} & x^3 & + & 3\dot{x}ox^2 & + & 3\dot{x}^2oox & + & \dot{x}^3o^3 \\ - & ax^2 & - & 2a\dot{x}ox & - & a\dot{x}^2oo & & \\ + & axy & + & a\dot{x}oy & + & a\dot{y}ox & + & a\dot{x}\dot{y}oo \\ - & y^3 & - & 3\dot{y}oy^2 & - & 3\dot{y}^2ooy & - & \dot{y}^3o^3 \end{array}\right\} = 0$$

Newton's method of fluxions

We seek the fluxion of the equation $x^3 - ax^2 + axy - y^3 = 0$

Substitute $x + \dot{x}o$ for x and $y + \dot{y}o$ for y, and expand:

$$\left.\begin{array}{cccccc} + & 3\dot{x}ox^2 & + & 3\dot{x}^2oox & + & \dot{x}^3o^3 \\ - & 2a\dot{x}ox & - & a\dot{x}^2oo & & \\ + & a\dot{x}oy & + & a\dot{y}ox & + & a\dot{x}\dot{y}oo \\ - & 3\dot{y}oy^2 & - & 3\dot{y}^2ooy & - & \dot{y}^3o^3 \end{array}\right\} = 0$$

Newton's method of fluxions

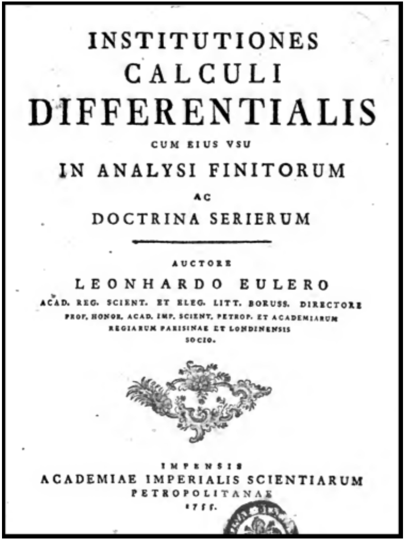
We are left with $3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3oo - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3oo = 0$

'But whereas o is supposed to be infinitely little, [...] the Terms that are multiply'd by it will be nothing in respect of the rest'

What remains is $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$

(Recall that we started with $x^3 - ax^2 + axy - y^3 = 0$)

Calculus systematised


INSTITUTIONES
CALCULI
DIFFERENTIALIS
CUM EIUS VSU
IN ANALYSI FINITORUM
AC
DOCTRINA SERIERUM

AUCTORE
LEONHARDO EULERO
ACAD. REG. SCIENT. ET ELEG. LITT. BORUSS. DIRECTORE
PROF. HONOR. ACAD. IMP. SCIENT. PETROP. ET ACADEMIARUM
REGIARUM PARISINAE ET LONDINENSIS
SOCIO.

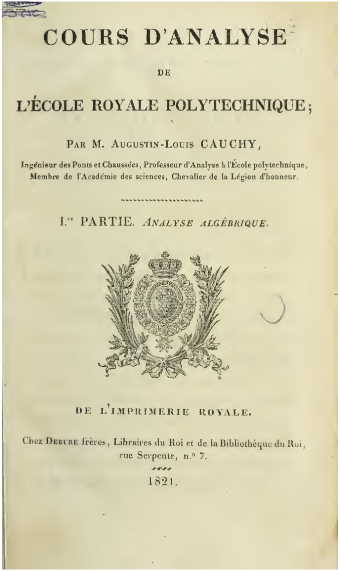
IMPENSIS
ACADEMIAE IMPERIALIS SCIENTIARUM
PETROPOLITANAE
1755.

Leonhard Euler, *Introductio in analysin infinitorum*, Lausanne, 1748

—, *Institutiones calculi differentialis*, St Petersburg, 1755

—, *Institutiones calculi integralis*, 3 vols, St Petersburg, 1768–1770

Cauchy's *Cours d'analyse* (1821)



COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

Ingénieur des Ponts et Chaussées, Professeur d'Analyse à l'École polytechnique, Membre de l'Académie des sciences, Chevalier de la Légion d'honneur.

I.re PARTIE. *ANALYSE ALGÉBRIQUE.*

DE L'IMPRIMERIE ROYALE.

Chez DEBURE frères, Libraires du Roi et de la Bibliothèque du Roi, rue Serpente, n.° 7.

1821.

Cauchy sequences

Cours d'analyse, pp. 124–125:

> *In order for the series* $u_0,\ u_1,\ u_2, \ldots$ [that is, $\sum u_i$] *to be convergent [...] it is necessary and sufficient that the partial sums*
>
> $$s_n = u_0 + u_1 + u_2 + \&c. \ldots + u_{n-1}$$
>
> *converge to a fixed limit* s*: in other words, it is necessary and sufficient that for infinitely large values of the number* n*, the sums*
>
> $$s_n,\ s_{n+1},\ s_{n+2},\ \&c. \ldots$$
>
> *differ from the limit* s*, and consequently from each other, by infinitely small quantities.*

Continuity

In *Cours d'analyse*, p. 34, Cauchy defined a function f to be continuous between certain limits if, for each x between those limits, the value of $f(x)$ is unique and finite, and $|f(x+\alpha)-f(\alpha)|$, where α is indefinitely small, decreases indefinitely with α.

In other words (p. 35): for x between the given limits, an infinitely small increase in x produces and infinitely small increase in $f(x)$

So Cauchy defined continuity on an interval, rather than at a point (and similarly elsewhere, when defining convergence)

He went on to derive basic results concerning continuous functions: that the composition of two continuous functions is continuous, the Intermediate Value Theorem, etc.

A theorem of Cauchy

Cauchy, *Cours d'analyse*, pp. 131–132:

> *When the various terms of a series are functions of a variable x, continuous with respect to this variable in the neighbourhood of a particular value for which the series is convergent, the sum s of the series is also, in the neighbourhood of this value, a continuous function of x.*

In other words: a convergent series of continuous functions converges to a continuous function.

Not true!

Cauchy's argument

Cauchy considered a sequence of continuous functions $u_0(x), u_1(x), u_2(x), \ldots$ on a given interval. He supposed that the corresponding series converges to a function $s(x)$. Partial sums are denoted by $s_n(x) = \sum_{j=0}^{n-1} u_n(x)$. The nth remainder term $r_n(x)$ is defined by $s(x) = s_n(x) + r_n(x)$.

Cauchy noted that each s_n is evidently continuous for values of x in the given interval. Suppose that we increase x by an infinitely small quantity α. For all values of n, the corresponding increase in $s_n(x)$ will also be infinitely small. For n very large ('très-considérable'), the increase in $r_n(x)$ becomes 'insensible'. Therefore, the increase in $s(x)$ can only be an infinitely small quantity.

Cauchy's argument

est convergente, la somme de cette série est représentée par

$$u_0 + u_1 + u_2 + u_3 + \&c. \ldots.$$

En vertu de cette convention, la valeur du nombre e se trouvera déterminée par l'équation

$$(6) \qquad e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.\ldots;$$

et, si l'on considère la progression géométrique

$$1,\ x,\ x^2,\ x^3,\ \&c.\ldots,$$

on aura, pour des valeurs numériques de x inférieures à l'unité,

$$(7) \qquad 1 + x + x^2 + x^3 + \&c.\ldots = \frac{1}{1-x}.$$

La série

$$u_0,\ u_1,\ u_2,\ u_3,\ \&c.\ldots$$

étant supposée convergente, si l'on désigne sa somme par s, et par s_n la somme de ses n premiers termes, on trouvera

$$s = u_0 + u_1 + u_2 + \ldots + u_{n-1} + u_n + u_{n+1} + \&c.\ldots$$
$$= s_n + u_n + u_{n+1} + \&c.\ldots,$$

et par suite

$$s - s_n = u_n + u_{n+1} + \&c.\ldots$$

De cette dernière équation il résulte que les quantités

$$u_n,\ u_{n+1},\ u_{n+2},\ \&c.\ldots.$$

formeront une nouvelle série convergente dont la somme sera équivalente à $s - s_n$. Si l'on représente cette même somme par r_n, on aura

$$s = s_n + r_n;$$

et r_n sera ce qu'on appelle le *reste* de la série (1) à partir du n.^{me} terme.

Lorsque, les termes de la série (1) renfermant une même variable x, cette série est convergente, et ses différens termes fonctions continues de x, dans le voisinage d'une valeur particulière attribuée à cette variable;

$$s_n,\quad r_n \text{ et } s$$

sont encore trois fonctions de la variable x, dont la première est évidemment continue par rapport à x dans le voisinage de la valeur particulière dont il s'agit. Cela posé, considérons les accroissemens que reçoivent ces trois fonctions, lorsqu'on fait croître x d'une quantité infiniment petite α. L'accroissement de s_n sera, pour toutes les valeurs possibles de n, une quantité infiniment petite; et celui de r_n deviendra insensible en même temps que r_n, si l'on attribue à n une valeur très-considérable. Par suite, l'accroissement de la fonction s ne pourra être qu'une quantité infiniment petite. De cette remarque on déduit immédiatement la proposition suivante.

1.^{er} THÉORÈME. *Lorsque les différens termes de la série (1) sont des fonctions d'une même variable x,*

A modern counterexample

For each $n \in \mathbb{N}$, define continuous functions f_n by

$$f_n(x) = \begin{cases} -1 & \text{if } x \leq -\frac{1}{n} \\ nx & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n} \\ +1 & \text{if } x \geq \frac{1}{n} \end{cases}$$

Now set $u_1(x) = f_1(x)$, and define new functions u_n recursively by

$$u_n(x) = f_n(x) - f_{n-1}(x)$$

Notice then that $s_n(x) = \sum_{j=1}^{n} u_j(x) = f_n(x)$

But we see that $s_n \to s$ as $n \to \infty$, where

$$s(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases}$$

which is discontinuous at $x = 0$

A modern counterexample

What happens to the remainders $r_n(x) = s(x) - s_n(x)$?

Outside the range $-\frac{1}{n} \leq x \leq \frac{1}{n}$, $r_n(x) = 0$, but inside:

$$r_n(x) = \begin{cases} -1 - nx & \text{if } -\frac{1}{n} \leq x < 0 \\ 0 & \text{if } x = 0 \\ 1 - nx & \text{if } 0 < x \leq \frac{1}{n} \end{cases}$$

For each x, $r_n(x) \to 0$ as $n \to \infty$, but this does not happen simultaneously for all values of x

Cauchy's remainders

Cauchy: For n very large, the increase in $r_n(x)$ becomes 'insensible'. But what does this mean?

One of the following modern statements? (Denoting Cauchy's interval by I)

$$\forall \varepsilon > 0 : \exists N : \forall x \in I : n > N \Rightarrow |r_n(x)| < \varepsilon$$

$$\forall \varepsilon > 0 : \forall x \in I : \exists N : n > N \Rightarrow |r_n(x)| < \varepsilon$$

The second is true for our modern counterexample, but the first is not — so there really is a distinction between the two

Cauchy clearly didn't make this distinction

εs and δs

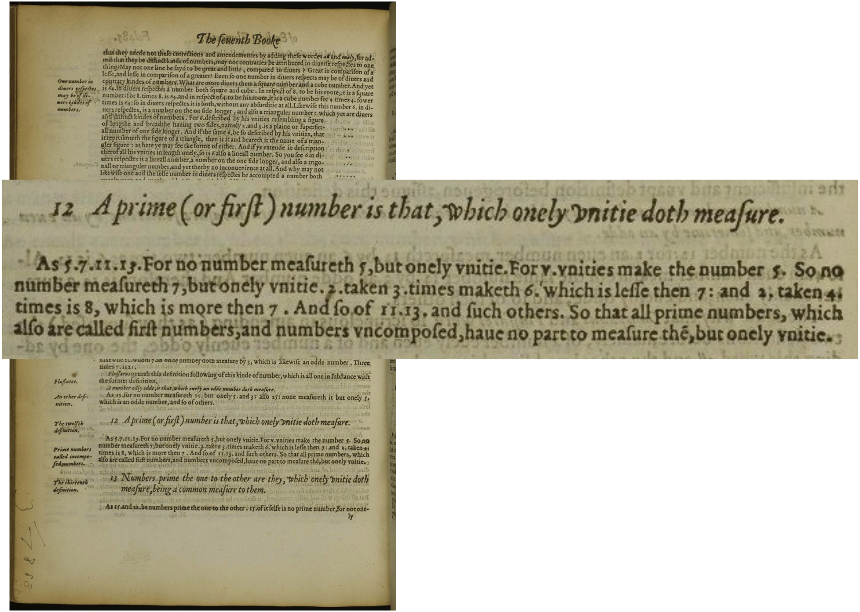
Karl Weierstrass (1815–1897)

Euclid's *Elements*

Euclid's *Elements*, in 13 books, compiled c. 250 BC

Books I–V:	definitions, postulates, plane geometry of lines and circles
Book VI:	similarity, proportion
Books VII–IX:	number theory
Book X:	commensurability, irrational numbers, surds
Books XI–XIII:	solid geometry ending with the classification of the regular polyhedra

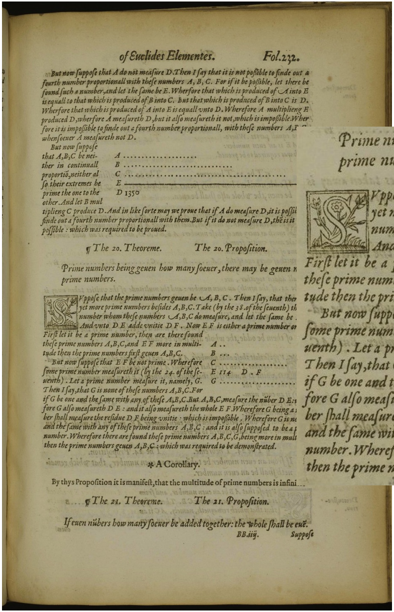
Euclid on prime numbers

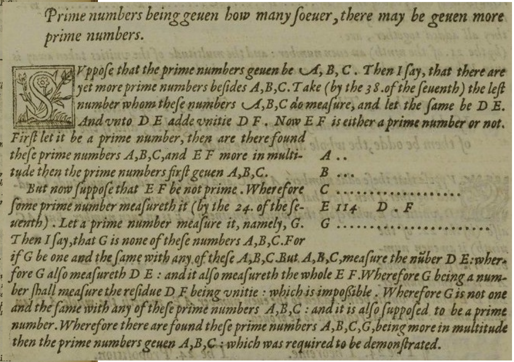


12 *A prime (or first) number is that, which onely vnitie doth measure.*

As 5.7.11.13. For no number measureth 5, but onely vnitie. For v. vnities make the number 5. So no number measureth 7, but onely vnitie. 2. taken 3. times maketh 6. which is lesse then 7: and 2. taken 4. times is 8, which is more then 7. And so of 11.13. and such others. So that all prime numbers, which also are called first numbers, and numbers vncomposed, haue no part to measure thē, but onely vnitie.

Euclid on prime numbers (Proposition IX.20)





Prime numbers being geuen how many soeuer, there may be geuen more prime numbers.

Suppose that the prime numbers geuen be A, B, C. Then I say, that there are yet more prime numbers besides A,B,C. Take (by the 38. of the seuenth) the lest number whom these numbers A,B,C do measure, and let the same be D E. And vnto D E adde vnitie D F. Now E F is either a prime number or not. First let it be a prime number, then are there found these prime numbers A,B,C, and E F more in multitude then the prime numbers first geuen A,B,C.

But now suppose that E F be not prime. Wherefore some prime number measureth it (by the 24. of the seuenth). Let a prime number measure it, namely, G. Then I say, that G is none of these numbers A,B,C. For if G be one and the same with any of these A,B,C. But A,B,C, measure the nũber D E: wherfore G also measureth D E : and it also measureth the whole E F. Wherefore G being a number shall measure the residue D F being vnitie : which is impossible. Wherefore G is not one and the same with any of these prime numbers A,B,C : and it is also supposed to be a prime number. Wherefore there are found these prime numbers A,B,C,G, being more in multitude then the prime numbers geuen A,B,C : which was required to be demonstrated.

A ..		
B ...		
C		
E 114	D . F	
G		

Number theory after Euclid

Very little for many centuries...

Diophantus' *Arithmetica* (13 books, *c.* AD 250) featured number problems; for example:

> Problem I.27: *Find two numbers such that their sum and product are given numbers*
>
> Problem III.19: *To find four numbers such that the square of their sum plus or minus any one singly gives a square*
>
> Problem V.9: *To divide unity into two parts such that, if a given number is added to either part, the result will be a square*

Restrictions on the permitted form of solutions to problems eventually gave rise to the notion of Diophantine equations

Number theory outside Europe

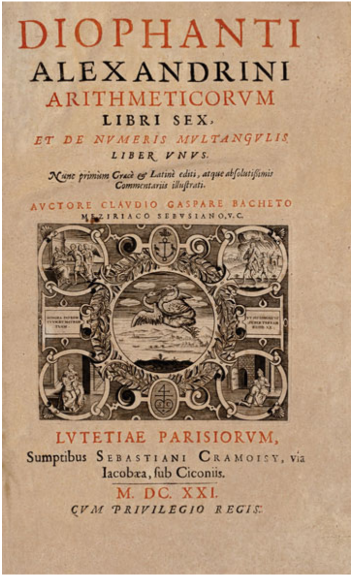
Sūnzǐ Suànjīng 孙子算经 (*The Mathematical Classic of Master Sun*) (3rd–5th century BC) contains a statement, but no proof, of the Chinese Remainder Theorem for the solution of simultaneous congruences

An algorithm for the solution was provided by Aryabhata in 6th-century India

In 7th-century India, Brahmagupta studied Diophantine equations (including Pell's equation $x^2 - Dy^2 = 1$)

These works were unknown in Europe until the 19th century

17th-century number theory



Bachet's Latin edition of Diophantus' *Arithmetica* (1621)

Pierre de Fermat owned a 1637 edition, which he studied and annotated

Fermat on number theory

Fermat's Little Theorem: if a is any integer and p is prime then p divides $a^p - a$

Conjectures on perfect numbers, and the search for Mersenne primes

Studies of Diophantine problems (more in a moment)

Published nothing — had to be exhorted to write his ideas down

The ‘Last Theorem’

Arithmetica Problem II.8 concerns the splitting of a given square number into two other squares

Fermat’s marginal note:

> *It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain.*

17th-century attitudes to number theory

Fermat failed to spark an interest in number theory in his contemporaries

Pascal to Fermat (1655):

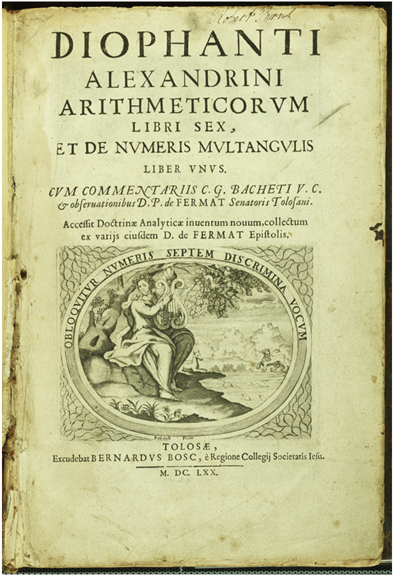
> *. . . seek elsewhere those who can follow you in your numerical discoveries . . . I confess to you that this goes far beyond me . . .*

Number-theoretic investigations were widely regarded as trivial and uninteresting

Huygens to Wallis:

> *There is no lack of better topics for us to spend our time on . . .*

The ‘rebirth’ of number theory



DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
CVM COMMENTARIIS C. G. BACHETI V. C.
& obſeruationibus D. P. de FERMAT Senatoris Toloſani.
Acceſſit Doctrinæ Analyticæ inuentum nouum, collectum
ex varijs eiuſdem D. de FERMAT Epiſtolis.

TOLOSÆ,
Excudebat BERNARDVS BOSC, è Regione Collegij Societatis Ieſu.
M. DC. LXX.

1670 edition of Bachet, published by Samuel Fermat, including his father’s notes

The ‘Last Theorem’ was not the only result for which Fermat failed to provide a proof

Number theory was ‘reborn’ from the attempts of Euler (and later Lagrange and Legendre) to fill the gaps left by Fermat

Euler on number theory

Euler (1747):

> *Nor is the author disturbed by the authority of the greatest mathematicians when they sometimes pronounce that number theory is altogether useless and does not deserve investigation. In the first place, knowledge is always good in itself, even when it seems to be far removed from common use. Secondly, all the aspects of the truth which are accessible to our mind are so closely related to one another that we dare not reject any of them as being altogether useless. [...] Moreover, even if the proof of some proposition does not appear to have any present use, it usually turns out that the method by which this problem has been solved opens the way to the discovery of more useful results.*

Euler on number theory

Euler (1747):

> *Consequently, the present author considers that he has by no means wasted his time and effort in attempting to prove various theorems concerning integers and their divisors. [...] Actually, far from being useless, this theory is of no little use even in analysis. Moreover, there is little doubt that the method used here by the author will turn out to be of no small value in other investigations of greater import.*

19th-century number theory

Gauss's *Disquisitiones arithmeticae* (1801) became a key text for many years to come: modular arithmetic, quadratic forms, cyclotomy, ...

Number-theoretic problems (especially attempts to prove Fermat's Last Theorem) led to the linking of number theory and abstract algebra in algebraic number theory

By the end of the 19th century, a new branch, analytic number theory, had also emerged (e.g., Riemann hypothesis, Prime Number Theory $\pi(x) \sim \frac{x}{\log x}$, ...)

Fermat's Last Theorem in the 19th century

Many special cases have been proved by Euler and others in the 18th century . . .

Sophie Germain proved the theorem for certain classes of primes

Gabriel Lamé claimed to have proved the theorem by factorising $x^n + y^n = z^n$ in a cyclotomic field $\mathbb{Q}(\omega_n)$ — but he assumed that the cyclotomic integers $\mathbb{Z}(\omega_n)$ factorise uniquely (which they don't)

Ernst Eduard Kummer tried to fix this by introducing ideal prime numbers — new elements adjoined to the number field which facilitate unique factorisation

Fermat's Last Theorem in the 19th century

For Kummer, an 'ideal' was an extra element adjoined to a number field, assumed to have the same divisibility properties as the original elements

If an ideal element divides original elements a and b, then it also divides $a \pm b$ and all multiples of a and b. So Kummer's 'ideal' gives rise to a subset of original elements which it divides

Richard Dedekind, on the other hand, simply took this subset (in fact, a submodule) as his notion of 'ideal'

For a general algebraic number field, Dedekind showed that every ideal may be decomposed uniquely as a product of (suitably defined) 'prime' ideals

The result was the development of ideal theory by Dedekind, Wolfgang Krull, Emmy Noether, and others

The nature of the History of Mathematics

> *... mathematics [rarely] progresses only by means of 'great and significant works' and 'substantial changes'. [...] the truth is far more subtle and far more interesting: mathematics is the result of a cumulative endeavour to which many people have contributed, and not only through their successes but through half-formed thoughts, tentative proposals, partially worked solutions, and even outright failure. No part of mathematics came to birth in the form that it now appears in a modern textbook: mathematical creativity can be slow, sometimes messy, often frustrating.*

Jacqueline A. Stedall, *From Cardano's great art to Lagrange's reflections: filling a gap in the history of algebra*, European Mathematical Society, 2011, p. ix